Homework 1: Sets and Measures

CR07: Selected Topics Information Theory (Fall 2019). ENS de Lyon
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1 Basic Operations with Sets

Proving the following statements is trivial. Often, the proof follows from the definition. Nonetheless, using the fact that $\mathcal{A} = \mathcal{B}$ if and only if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$ provides a formal element of proof.

1. Proof of Theorem 1.6. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be some sets. Prove the following identities

    - $\mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A}$ and $\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A}$ (Commutative Property)
    - $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} = (\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C} = \mathcal{A} \cup (\mathcal{B} \cup \mathcal{C})$ and
      $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C} = (\mathcal{A} \cap \mathcal{B}) \cap \mathcal{C} = \mathcal{A} \cap (\mathcal{B} \cap \mathcal{C})$ (Associative Property)
    - $(\mathcal{A} \cup \mathcal{B}) \cap \mathcal{C} = (\mathcal{A} \cap \mathcal{C}) \cup (\mathcal{B} \cap \mathcal{C})$ and
      $(\mathcal{A} \cap \mathcal{B}) \cup \mathcal{C} = (\mathcal{A} \cup \mathcal{C}) \cap (\mathcal{B} \cup \mathcal{C})$ (Distributive Property)
    - $\mathcal{A} \cap \mathcal{A} = \mathcal{A} \cup \mathcal{A} = \mathcal{A}$. (Idempotent Property)

2. Proof of Theorem 1.8. Given a non-empty subset $\mathcal{A}$ of a universal set $\mathcal{O}$, prove that $a \in \mathcal{A}$ implies $a \notin \mathcal{A}^{\mathsf{c}}$.

3. Proof of Theorem 1.9. Given two sets $\mathcal{A}$ and $\mathcal{B}$ of a universal set $\mathcal{O}$, such that $\mathcal{A} \subseteq \mathcal{B}$, prove that $\mathcal{A}^{\mathsf{c}} \supseteq \mathcal{B}^{\mathsf{c}}$.

4. Proof of Theorem 1.10. Given two subsets $\mathcal{A}$ and $\mathcal{B}$ of a universal set $\mathcal{O}$, prove that
$$\mathcal{A} \setminus \mathcal{B} = \mathcal{A} \cap \mathcal{B}^{\mathsf{c}}. \tag{1}$$

5. Proof of Theorem 1.15. Let $\mathcal{A}$ and $\mathcal{B}$ be two sets. Prove that
$$\mathcal{A} \cup \mathcal{B} = (\mathcal{A}^{\mathsf{c}} \cap \mathcal{B}^{\mathsf{c}})^{\mathsf{c}} \text{ and} \tag{2}$$
$$\mathcal{A} \cap \mathcal{B} = (\mathcal{A}^{\mathsf{c}} \cup \mathcal{B}^{\mathsf{c}})^{\mathsf{c}}. \tag{3}$$

2 Algebraic Structures

1. Let $\mathscr{F}$ and $\mathscr{G}$ be two σ-fields of $\mathcal{O}$. Prove that, $\mathscr{F} \cap \mathscr{G}$ is also a σ-field of $\mathcal{O}$.

2. Let $\mathscr{F}$ and $\mathscr{G}$ be two σ-fields of $\mathcal{O}$. Provide an example to show that $\mathscr{F} \cup \mathscr{G}$ is not necessarily a σ-field of $\mathcal{O}$.

3 Measures

1. Proof of Theorem 2.4. Let $(\mathcal{A}, \mathscr{F})$ and $(\mathcal{B}, \mathscr{G})$ be two measurable spaces, such that $\mathscr{G} = \sigma(\mathcal{C})$, for some set of subsets $\mathcal{C}$. Prove that, a function $f : \mathcal{A} \to \mathcal{B}$ is measurable relative to $(\mathcal{A}, \mathscr{F})$ and $(\mathcal{B}, \mathscr{G})$ if for all $\mathcal{G} \in \mathcal{C}$,
$$f^{-1}(\mathcal{G}) \in \mathscr{F}. \tag{4}$$

2. Proof of Theorem 2.5. Consider a measurable function f with respect to $(\mathcal{A}, \mathscr{E})$ and $(\mathcal{B}, \mathscr{F})$. Consider also a measurable function g with respect to $(\mathcal{B}, \mathscr{F})$ and $(\mathcal{C}, \mathscr{G})$. Prove that the composition $f \circ g$ is measurable with respect to $(\mathcal{A}, \mathscr{E})$ and $(\mathcal{C}, \mathscr{G})$.

3. Proof of Theorem 2.11. Let μ be a measure on the σ-field $\mathscr{F}$. Prove that
   (a) $\mu(\emptyset) = 0$;
   (b) $\forall (\mathcal{A}, \mathcal{B}) \in \mathscr{F}^2$, $\mu(\mathcal{A} \cup \mathcal{B}) + \mu(\mathcal{A} \cap \mathcal{B}) = \mu(\mathcal{A}) + \mu(\mathcal{B})$;
   (c) $\forall (\mathcal{A}, \mathcal{B}) \in \mathscr{F}^2$, with $\mathcal{A} \subset \mathcal{B}$, $\mu(\mathcal{B}) = \mu(\mathcal{A}) + \mu(\mathcal{B} \setminus \mathcal{A})$.

4. Proof of Theorem 2.12. Consider a σ-field $\mathscr{F}$ on a set $\mathcal{O}$ and let μ be a measure on $\mathscr{F}$. Consider also an infinite sequence of subsets $\mathcal{A}_1, \mathcal{A}_2, \ldots,$ in $\mathscr{F}$. Prove that
   (a) if $\mathcal{A}_n \uparrow \mathcal{A}$, $\lim_{n\to\infty} \mu(\mathcal{A}_n) = \mu(\mathcal{A})$; and
   (b) if $\mathcal{A}_n \downarrow \mathcal{A}$ and $\mu(\mathcal{O}) < \infty$, $\lim_{n\to\infty} \mu(\mathcal{A}_n) = \mu(\mathcal{A})$.

5. Proof of Theorem 2.14. Let f be an arbitrary Borel measurable function on $(\mathcal{O}, \mathscr{F})$. Prove that the functions f^+ and f^- are both Borel measurable functions on $(\mathcal{O}, \mathscr{F})$.

4 Radom-Nikodym Derivatives

1. Proof of Theorem 2.20. Given a mesurable space $(\mathcal{O}, \mathscr{F})$ and a non-negative Borel measurable function $f : \mathcal{O} \to \mathbb{R}$ with respect to $(\mathcal{O}, \mathscr{F})$, let $\nu : \mathscr{F} \to \mathbb{R}_+$ be
$$\nu(\mathcal{A}) = \int_{\mathcal{A}} f \mathrm{d}\mu. \tag{5}$$
Prove that, ν is a measure on $(\mathcal{O}, \mathscr{F})$.

2. Proof of Theorem 2.22. Let μ and ν be two measures on a given measurable space $(\mathcal{O}, \mathscr{F})$ with ν being absolutely continuous with respect to μ and μ being σ-finite. Prove that

   - for all $x \in \mathcal{O}$, $\frac{\mathrm{d}\mu}{\mathrm{d}\mu}(x) = 1$.
   - if $f : \mathcal{O} \to \mathbb{R}_+$ is a non-negative Borel measurable function with respect to $(\mathcal{O}, \mathscr{F})$, it holds that for all $\mathcal{A} \in \mathscr{F}$,
   $$\int_{\mathcal{A}} f \mathrm{d}\nu = \int_{\mathcal{A}} f \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \mathrm{d}\mu; \tag{6}$$
   - if λ is a σ-finite measure on $(\mathcal{O}, \mathscr{F})$, μ is absolutely continuous with respect to λ, it holds that for all $x \in \mathcal{O}$
   $$\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}(x) = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x); \text{ and} \tag{7}$$
   - if μ is absolutely continuous with respect to ν, and ν is σ-finite, it holds that for all $x \in \mathcal{O}$
   $$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x) = 1. \tag{8}$$

3. Proof of Theorem 2.23. Let μ be a σ-finite measure on $(\mathcal{O}, \mathscr{F})$ and $\nu_1, \nu_2, \ldots, \nu_n$ be finite measures on $(\mathcal{O}, \mathscr{F})$ such that for all $k \in \{1, 2, \ldots, n\}$, ν_k is absolutely continuous with μ. Prove that for all $x \in \mathcal{O}$,
$$\frac{\mathrm{d} \sum_{t=1}^{n} \nu_t}{\mathrm{d}\mu}(x) = \sum_{t=1}^{m} \frac{\mathrm{d}\nu_t}{\mathrm{d}\mu}(x). \tag{9}$$

Moreover, if ν is a measure on $(\mathcal{O}, \mathscr{F})$ such that for all $\mathcal{A} \in \mathscr{F}$, $\nu(\mathcal{A}) = \lim_{n\to\infty} \sum_{t=1}^{n} \nu_t(\mathcal{A})$, prove that ν is absolutely continuous with μ and

$$\lim_{n\to\infty} \frac{\mathrm{d}\sum_{t=1}^{n} \nu_t}{\mathrm{d}\mu}(x) = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x). \tag{10}$$