Duality with Linear-Feedback Schemes for the Scalar Gaussian MAC and BC

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joint work with
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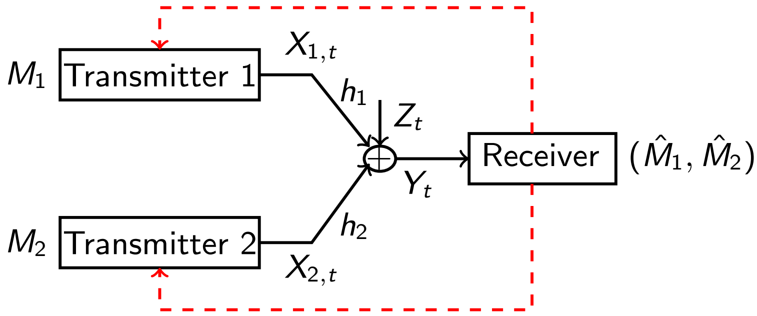
Contribution

Duality with linear-feedback schemes for scalar Gaussian MAC and BC.

$$\mathcal{C}^{\text{linfb}}_{\text{MAC}}(h_1, h_2; P) = \mathcal{C}^{\text{linfb}}_{\text{BC}}(h_1, h_2; P).$$

- ☺ $\mathcal{C}^{\text{linfb}}_{\text{MAC}}$ is well-known (Ozarow'84)
- $\Rightarrow$ Can characterize $\mathcal{C}^{\text{linfb}}_{\text{BC}}$.
- $\Rightarrow$ Identify best linear-feedback scheme for scalar Gaussian BC.

2-user Scalar Gaussian MAC with Feedback

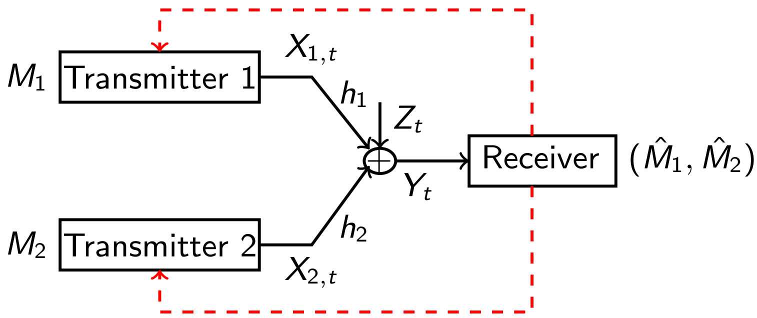


- Messages M_1 and M_2 independent ; $M_i \sim \mathcal{U}\{1, \dots, 2^{nR_i}\}$

$$Y_t = h_1 x_{1,t} + h_2 x_{2,t} + Z_t, \quad t \in \{1, \dots, n\}$$

- $h_1, h_2 \neq 0$ constant channel coefficients
- $\{Z_t\}$ i.i.d.$\sim \mathcal{N}(0,1)$
- Sum-power constraint: $\frac{1}{n} \sum_{t=1}^{n} \left(\mathrm{E}\left[X_{1,t}^2\right] + \mathrm{E}\left[X_{2,t}^2\right] \right) \leq P$

2-user Scalar Gaussian MAC with Feedback



- Perfect output feedback:

$$X_{i,t} = \varphi_{i,t}^{(n)}(M_i, Y_1, \dots, Y_{t-1}), \quad t \in \{1, \dots, n\}, \quad i \in \{1, 2\}.$$

Linear-Feedback coding schemes (LFCS) for MAC

$$\mathbf{X}_i \triangleq \begin{pmatrix} X_{i,1} \\ \vdots \\ X_{i,n} \end{pmatrix}, \qquad \mathbf{Y} \triangleq \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}, \qquad \mathbf{V}_i \triangleq \varphi_i^{(n)}(M_i) = \begin{pmatrix} V_{i,1} \\ \vdots \\ V_{i,n} \end{pmatrix}.$$

- Feedback used linearly:

$$\mathbf{X}_i = \mathbf{V}_i + \mathsf{C}_i \mathbf{Y}, \quad i \in \{1,2\}.$$

- Causal feedback $\Rightarrow$ $\mathsf{C}_1, \mathsf{C}_2$: strictly lower-triangular matrices.
- $\mathcal{C}_{\text{MAC}}^{\text{linfb}}$ = all rates achievable with LFCSs over MAC.
- $C_{\text{MAC},\Sigma}^{\text{linfb}}$ = largest sum-rate achieved with LFCSs over MAC.

Linear-Feedback capacity for MAC

- Ozarow'84: feedback capacity region under individual power constraints

$$\bigcup_{\rho \in [0,1]} \left\{ (R_1, R_2) : \begin{aligned} R_1 &\leq \frac{1}{2} \log\left(1 + h_1^2 P_1 (1-\rho^2)\right), \\ R_2 &\leq \frac{1}{2} \log\left(1 + h_2^2 P_2 (1-\rho^2)\right), \\ R_1 + R_2 &\leq \frac{1}{2} \log\left(1 + h_1^2 P_1 + h_2^2 P_2 + 2\sqrt{h_1^2 h_2^2 P_1 P_2}\rho\right). \end{aligned} \right\}$$

Linear-Feedback capacity for MAC

- Ozarow'84: feedback capacity region under individual power constraints
- under a sum-power constraint P

$$\mathcal{C}^{\text{fb}}_{\text{MAC}}(h_1, h_2; P) =$$

$$\bigcup_{\substack{P_1,P_2\geq 0:\\ P_1+P_2=P}} \bigcup_{\rho\in[0,1]} \left\{ (R_1,R_2): \begin{array}{rl} R_1 & \leq \frac{1}{2}\log\left(1+h_1^2P_1(1-\rho^2)\right), \\ R_2 & \leq \frac{1}{2}\log\left(1+h_2^2P_2(1-\rho^2)\right), \\ R_1+R_2 & \leq \frac{1}{2}\log\left(1+h_1^2P_1+h_2^2P_2+2\sqrt{h_1^2h_2^2P_1P_2}\rho\right). \end{array} \right\}$$

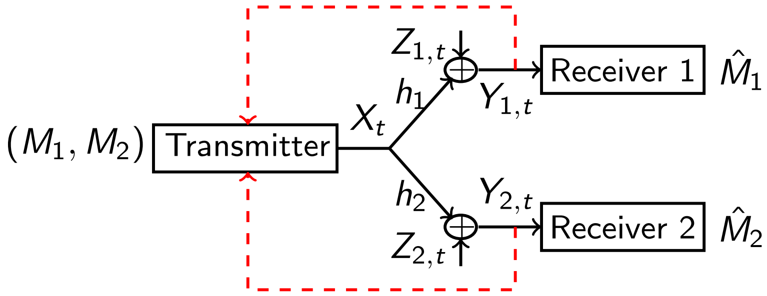
Linear-Feedback capacity for MAC

- Ozarow'84: feedback capacity region under individual power constraints
- under a sum-power constraint P
- Ozarow's scheme is a LFCS, based on LMMSE estimates

$$\mathcal{C}^{\text{fb}}_{\text{MAC}}(h_1, h_2; P) = \mathcal{C}^{\text{linfb}}_{\text{MAC}}(h_1, h_2; P) =$$

$$\bigcup_{\substack{P_1,P_2\geq 0:\\ P_1+P_2=P}} \bigcup_{\rho\in[0,1]} \left\{ (R_1,R_2): \begin{array}{rl} R_1 & \leq \frac{1}{2}\log\left(1+h_1^2P_1(1-\rho^2)\right), \\ R_2 & \leq \frac{1}{2}\log\left(1+h_2^2P_2(1-\rho^2)\right), \\ R_1+R_2 & \leq \frac{1}{2}\log\left(1+h_1^2P_1+h_2^2P_2+2\sqrt{h_1^2h_2^2P_1P_2}\rho\right). \end{array} \right\}$$

2-user Scalar Gaussian BC with Feedback

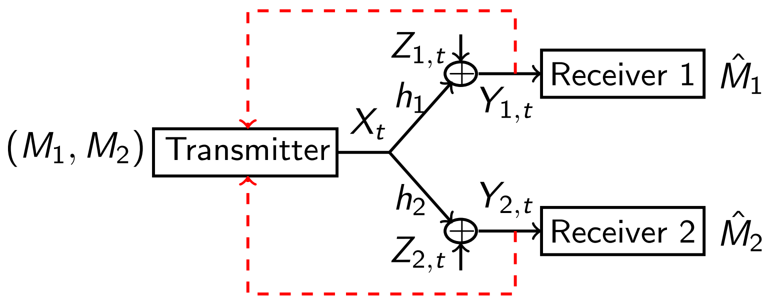


- Messages M_1 and M_2 independent ; $M_i \sim \mathcal{U}\{1, \ldots, 2^{nR_i}\}$

$$Y_{i,t} = h_i x_t + Z_{i,t}, \quad i \in \{1, 2\}, \quad t \in \{1 \ldots, n\}$$

- $h_1, h_2 \neq 0$ constant channel coefficients
- Independent $\{Z_{1,t}\}_{t=1}^{n}$ and $\{Z_{2,t}\}_{t=1}^{n}$ i.i.d.$\sim \mathcal{N}(0, 1)$
- Power constraint: $\frac{1}{n} \sum_{t=1}^{n} \mathrm{E}\left[X_t^2\right] \leq P$

2-user Scalar Gaussian BC with Feedback



- Perfect output feedback:

$$X_t = \varphi_t^{(n)}(M_1, M_2, Y_{1,1}, \ldots, Y_{1,t-1}, Y_{2,1}, \ldots, Y_{2,t-1}).$$

Linear-Feedback coding schemes (LFCS) for BC

$$\mathbf{X} \triangleq \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}, \qquad \mathbf{Y}_i \triangleq \begin{pmatrix} Y_{i,1} \\ \vdots \\ Y_{i,n} \end{pmatrix}, \qquad \mathbf{W} \triangleq \xi^{(n)}(M_1, M_2) = \begin{pmatrix} W_1 \\ \vdots \\ W_n \end{pmatrix}.$$

- Feedback used linearly:

$$\mathbf{X} = \mathbf{W} + \mathsf{A}_1 \mathbf{Y}_1 + \mathsf{A}_2 \mathbf{Y}_2$$

- Causal feedback $\Rightarrow$ $\mathsf{A}_1, \mathsf{A}_2$ **strictly lower-triangular** matrices.
- $\mathcal{C}_{\text{BC}}^{\text{linfb}}$ = all rates achievable with LFCS over BC.
- $C_{\text{BC},\Sigma}^{\text{linfb}}$ = largest sum-rate achievable with LFCS over BC.

Achievable regions for BC

- Feeback Capacity is open!!
- Linear-feedback schemes:
  - Ozarow-Leung'84,
  - Kramer'02,
  - Gastpar/Lapidoth/Steinberg/Wigger'11,
  - Ardestanizadeh/Minero/Franceschetti'12 ...
- Non-linear feedback schemes:
  - Shayewitz-Wigger'13,
  - Venkataramanan/Pradhan'11,
  - Wu/Wigger'13

- Best achievable regions by LFCS.

Previous Duality Results

- (Ardestanizadeh, Minero, Franceschetti'12) for BC:
  - based on LQG control theory (not on LMMSEs!)
  - achieves MAC-sum-capacity when $h_1 = h_2$ (over BC)

No feedback duality:
(Vishwanath/Jindal/Goldsmith'03, Viswanath/Tse'03, Weingarten/Steinberg/Shamai'06)

$$\mathcal{C}_{\mathrm{MAC}}^{\mathrm{nofb}}(\mathsf{H}_1^{\mathsf{T}}, \mathsf{H}_2^{\mathsf{T}}; P) = \mathcal{C}_{\mathrm{BC}}^{\mathrm{nofb}}(\mathsf{H}_1, \mathsf{H}_2; P)$$

Main results

Duality with Linear-Feedback Schemes for the Scalar Gaussian MAC and BC

$$\mathcal{C}_{\mathrm{MAC}}^{\mathrm{linfb}}(h_1, h_2; P) = \mathcal{C}_{\mathrm{BC}}^{\mathrm{linfb}}(h_1, h_2; P).$$

From Ozarow'84:

$$\mathcal{C}_{\mathrm{BC}}^{\mathrm{linfb}}(h_1, h_2; P) = \mathcal{C}_{\mathrm{MAC}}^{\mathrm{linfb}}(h_1, h_2; P) =$$

$$\bigcup_{\substack{P_1,P_2\geq 0:\\ P_1+P_2=P}} \bigcup_{\rho\in[0,1]} \left\{ (R_1,R_2): \begin{array}{rl} R_1 & \leq \frac{1}{2}\log\left(1+h_1^2P_1(1-\rho^2)\right), \\ R_2 & \leq \frac{1}{2}\log\left(1+h_2^2P_2(1-\rho^2)\right), \\ R_1+R_2 & \leq \frac{1}{2}\log\left(1+h_1^2P_1+h_2^2P_2+2\sqrt{h_1^2h_2^2P_1P_2}\rho\right). \end{array} \right\}$$

Main Results

Corollary on Linear-feedback sum-capacities

$C^{\text{linfb}}_{\text{BC},\Sigma}(h_1, h_2; P) = C^{\text{linfb}}_{\text{MAC},\Sigma}(h_1, h_2; P) =$

$$\sup_{\substack{P_1, P_2 \geq 0: \\ P_1 + P_2 = P}} \frac{1}{2} \log \left(1 + h_1^2 P_1 + h_2^2 P_2 + 2\sqrt{h_1^2 h_2^2 P_1 P_2} \cdot \rho^\star(h_1, h_2; P_1, P_2) \right)$$

- $\rho^\star(h_1, h_2; P_1, P_2)$ is the unique solution in $[0, 1]$ to

$$(1 + h_1^2 P_1 + h_2^2 P_2 + 2\sqrt{h_1^2 h_2^2 P_1 P_2 \rho}) = (1 + h_1^2 P_1(1-\rho^2))(1 + h_2^2 P_2(1-\rho^2))$$

If $h_1 = h_2 = h$:

$$C^{\text{linfb}}_{\text{BC},\Sigma}(h, h; P) = \frac{1}{2} \log \left(1 + h^2 P + h^2 P \cdot \rho^\star(h, h; P, P)\right)$$

$\Rightarrow$ (Ardestanizadeh, Minero, Franceschetti'12) is optimal.

Main Results

Comparison to Other Bounds for BC

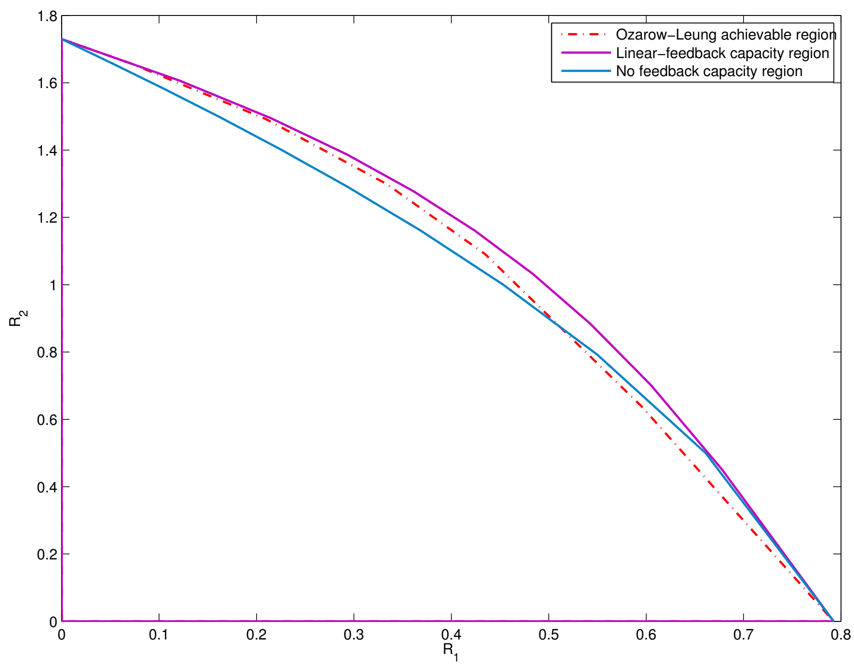


Figure: $h_1 = \frac{1}{\sqrt{5}}$, $h_2 = 1$ and power $P = 10$.

Sketch of the Proof

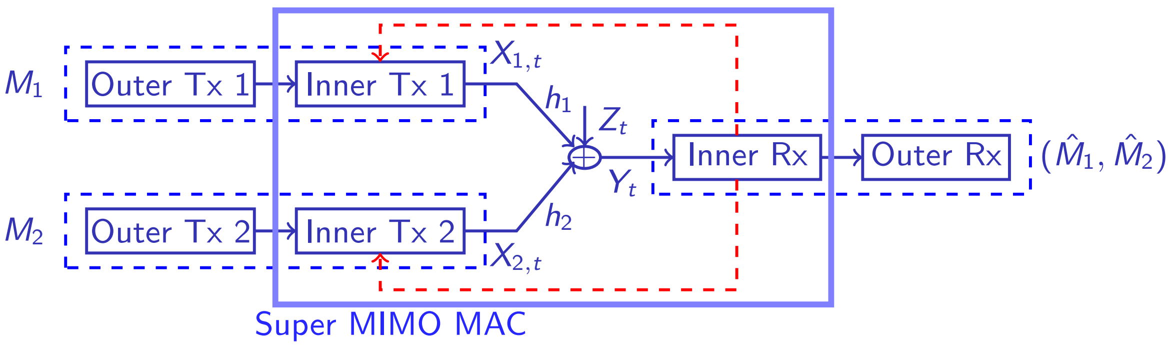
1. A class of LFCSs for MAC/BC achieving $\mathcal{C}^{\text{linfb}}_{\text{MAC}}(h_1, h_2; P)/\mathcal{C}^{\text{linfb}}_{\text{BC}}(h_1, h_2; P)$

$\rightarrow$ multiletter-expressions for $\mathcal{C}^{\text{linfb}}_{\text{MAC}}(h_1, h_2; P)$ and $\mathcal{C}^{\text{linfb}}_{\text{BC}}(h_1, h_2; P)$

2. Identify pairs of MAC/BC parameters that achieve the same rate-regions over MAC and BC:

$\rightarrow$ For each MAC-params, $\exists$ BC-params achieving the same region, vice-versa.

Structure of Optimal LFCSs for MAC



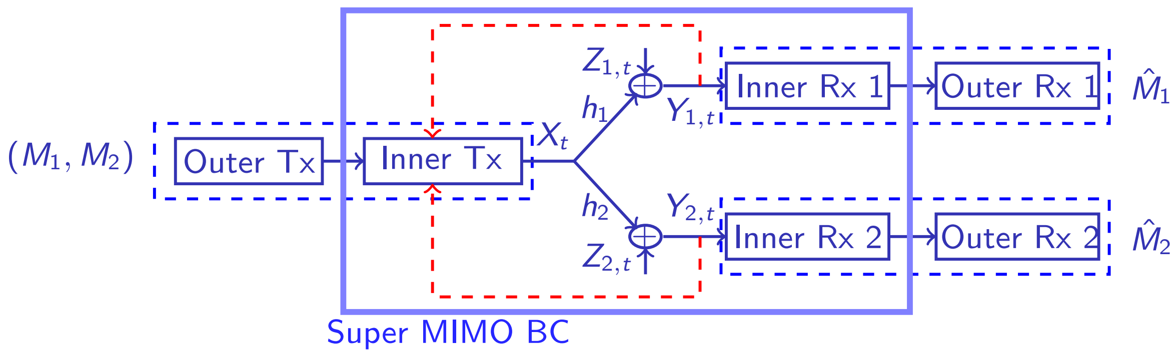
- Prameters: $\eta \in \mathbb{N}$, $\mathsf{C}_1^{(\eta)}, \mathsf{C}_2^{(\eta)}$: strictly lower-triangular matrices
- Divide the blocklength into subblocks of length-η
- Inner Code:
  - uses feedback linearly: LFCS$(\eta, \mathsf{C}_1^{(\eta)}, \mathsf{C}_2^{(\eta)})$
  - η uses of Scalar MAC = 1 use of a Super MIMO MAC.
- Outer Code:
  - ignores feedback,
  - codes for $\mathcal{C}^{\text{nofb}}_{\text{MAC}}(\text{Super MIMO MAC}) \triangleq \mathcal{R}_{\text{MAC}}(\eta, \mathsf{C}_1^{(\eta)}, \mathsf{C}_2^{(\eta)}, h_1, h_2; P)$,
- achieves $\frac{1}{\eta}\mathcal{R}_{\text{MAC}}(\eta, \mathsf{C}_1^{(\eta)}, \mathsf{C}_2^{(\eta)}, h_1, h_2; P)$ over Scalar MAC.

Multiletter-expression for $\mathcal{C}_{\mathrm{MAC}}^{\mathrm{linfb}}$

$$\mathcal{C}_{\mathrm{MAC}}^{\mathrm{linfb}}(h_1, h_2; P) = \mathrm{cl}\left(\bigcup_{\eta, \mathsf{C}_1^{(\eta)}, \mathsf{C}_2^{(\eta)}} \frac{1}{\eta} \mathcal{R}_{\mathrm{MAC}}(\eta, \mathsf{C}_1^{(\eta)}, \mathsf{C}_2^{(\eta)}, h_1, h_2; P) \right)$$

- η: positive integer
- $\mathsf{C}_1^{(\eta)}, \mathsf{C}_2^{(\eta)}$: strictly lower-triangular η-by-η matrices

Structure of Optimal LFCSs for BC



- Inner Code $(\eta, \mathsf{A}_1^{(\eta)}, \mathsf{A}_2^{(\eta)})$:
  - uses feedback linearly.
  - η uses of Scalar BC $=$ 1 use of a Super MIMO BC.
- Outer Code:
  - ignores feedback,
  - codes for $\mathcal{C}_{\mathrm{BC}}^{\mathrm{nofb}}$(Super MIMO BC) $\triangleq \mathcal{R}_{\mathrm{BC}}(\eta, \mathsf{A}_1^{(\eta)}, \mathsf{A}_2^{(\eta)}, h_1, h_2; P)$,
- achieves $\frac{1}{\eta}\mathcal{R}_{\mathrm{BC}}(\eta, \mathsf{A}_1^{(\eta)}, \mathsf{A}_2^{(\eta)}, h_1, h_2; P)$ over Scalar BC.

Multiletter-expression for $\mathcal{C}_{\text{BC}}^{\text{linfb}}$

$$\mathcal{C}_{\text{BC}}^{\text{linfb}}(h_1, h_2; P) = \text{cl}\left(\bigcup_{\eta, \text{A}_1^{(\eta)}, \text{A}_2^{(\eta)}} \frac{1}{\eta} \mathcal{R}_{\text{BC}}(\eta, \text{A}_1^{(\eta)}, \text{A}_2^{(\eta)}, h_1, h_2; P) \right),$$

- η: positive integer
- $\text{A}_1^{(\eta)}, \text{A}_2^{(\eta)}$:strictly lower-triangular η-by-η matrices.

Pairs of Dual MAC/BC Schemes

Pairs of Dual MAC/BC Schemes

$$\text{A}_i^{(\eta)} = \bar{\text{C}}_i^{(\eta)} \implies \mathcal{R}_{\text{MAC}}(\eta, \text{C}_1^{(\eta)}, \text{C}_2^{(\eta)}, h_1, h_2; P) = \mathcal{R}_{\text{BC}}(\eta, \text{A}_1^{(\eta)}, \text{A}_2^{(\eta)}, h_1, h_2; P)$$

- $\bar{\text{C}}_i^{(\eta)} \triangleq$ *reverse image* of $\text{C}_i^{(\eta)}$ (mirror image w.r.t. the couter-diagonal)
- Ozarow's optimal MAC scheme:

$$\text{C}_i^{(\eta)} \text{ Toeplitz} \quad \Rightarrow \quad \bar{\text{C}}_i^{(\eta)} = \text{C}_i^{(\eta)}$$

Optimal BC scheme:

$$\text{A}_i^{(\eta)} = \text{C}_i^{(\eta)}, \quad i \in \{1, 2\}$$

Extension to MAC and BC with $K \geq 2$-users

Duality with LFCSs for the Scalar Gaussian MAC and BC with $K \geq 2$-users

$$\mathcal{C}^{\text{linfb}}_{\text{MAC}}(h_1, \ldots, h_K; P) = \mathcal{C}^{\text{linfb}}_{\text{BC}}(h_1, \ldots, h_K; P).$$

If $h_1 = \cdots = h_K = h$,
(Kramer'02, Ardestanizadeh/Wigger/Kim/Javidi'12, a symmetry argument)

$$\mathcal{C}^{\text{linfb}}_{\text{BC},\Sigma}(h, \ldots, h; P) = \mathcal{C}^{\text{linfb}}_{\text{MAC},\Sigma}(h, \ldots, h; P) = \frac{1}{2}\log\left(1 + P\phi(K, P)\right),$$

where $\phi(K, P)$ is the unique solution in $[1, K]$ to:

$$(1 + P\phi)^{K-1} = \left(1 + \frac{P}{K}\phi(K - \phi)\right).$$

$\Rightarrow$ (Ardestanizadeh, Minero, Franceschetti'12) is optimal for $K \geq 3$-user sym. BC

Summary

Duality with Linear-Feedback Schemes for the Scalar Gaussian MAC and BC

$$\mathcal{C}^{\text{linfb}}_{\text{MAC}}(h_1, h_2; P) = \mathcal{C}^{\text{linfb}}_{\text{BC}}(h_1, h_2; P).$$

- Extends to scalar MAC and BC with arbitrary $K \geq 2$-users
- Extends also to scalar MAC and BC with one-sided feedback
- Multi-letter expressions for $\mathcal{C}^{\text{linfb}}_{\text{MAC}}$ and $\mathcal{C}^{\text{linfb}}_{\text{BC}}$
- Single-letter characterization of $\mathcal{C}^{\text{linfb}}_{\text{BC}}$ and $\mathcal{C}^{\text{linfb}}_{\text{BC},\Sigma}$