MAC-BC Duality with Linear-Feedback Schemes

Selma Belhadj Amor {belhadjamor@telecom-paristech.fr}

TELECOM ParisTech

joint work with
Yossef Steinberg and Michèle Wigger

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Contribution

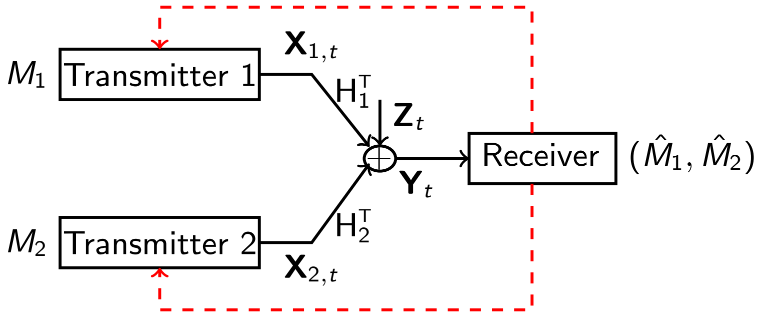
Multi-Antenna MAC-BC Duality with Linear-Feedback Schemes

$$\mathcal{C}_{\text{MAC}}^{\text{linfb}}(\mathsf{H}_1^{\mathsf{T}}, \mathsf{H}_2^{\mathsf{T}}; P) = \mathcal{C}_{\text{BC}}^{\text{linfb}}(\mathsf{H}_1, \mathsf{H}_2; P)$$

- Regions for BC with "specific" linear-feedback schemes
  - ☹ Optimization step to determine $\mathcal{C}_{\text{BC}}^{\text{linfb}}$ is very difficult
- Transfer MAC results to BC
  - ☺ $\mathcal{C}_{\text{MAC,SISO}}^{\text{linfb}}$, $\mathcal{C}_{\text{MAC,SIMO}}^{\text{linfb}}$, and $\mathcal{C}_{\text{MAC,MISO}}^{\text{linfb}}$ are known

    (Ozarow'84) & (Jafar/Goldsmith'06)
  - ⇒ Obtain $\mathcal{C}_{\text{BC,SISO}}^{\text{linfb}}$, $\mathcal{C}_{\text{BC,SIMO}}^{\text{linfb}}$, and $\mathcal{C}_{\text{BC,MISO}}^{\text{linfb}}$ (skip optimization step!)
- Tells us how to code over Gaussian BC when using feedback linearly

2-user MIMO Gaussian MAC with Feedback



$$\mathbf{Y}_t = H_1^T \mathbf{x}_{1,t} + H_2^T \mathbf{x}_{2,t} + \mathbf{Z}_t, \quad t \in \{1, \dots, n\}$$

- $H_i \in \mathbb{R}^{\nu_i \times \kappa}$, deterministic
- $\mathbf{x}_{i,t} \in \mathbb{R}^{\nu_i}$, $\mathbf{Y}_t \in \mathbb{R}^{\kappa}$
- $\{\mathbf{Z}_t\}$ i.i.d.$\sim \mathcal{N}(0, I)$
- Messages M_1 and M_2 independent ; $M_i \sim \mathcal{U}\{1, \dots, 2^{nR_i}\}$
- Sum-power constraint: $\frac{1}{n} \sum_{t=1}^{n} \left(E\left[\|\mathbf{X}_{1,t}\|^2\right] + E\left[\|\mathbf{X}_{2,t}\|^2\right] \right) \leq P$

Linear-Feedback Coding Schemes (LFCS) for MIMO MAC

- Feedback is used linearly

$$\mathbf{X}_{i,t} = \mathbf{U}_{i,t} + \sum_{\tau=1}^{t-1} C_{i,\tau,t} \mathbf{Y}_\tau, \quad t \in \{1, \dots, n\}, \quad i \in \{1, 2\}$$

  - $C_{i,\tau,t}$: arbitrary ν_i-by-κ matrices
  - $\mathbf{U}_{i,t} \triangleq \varphi_{i,t}^{(n)}(M_i), \quad t \in \{1, \dots, n\}$
- $\varphi_{i,t}^{(n)}$ and decoding can be arbitrary
- $\mathcal{C}_{\text{MAC}}^{\text{linfb}}$ = all rates achievable with LFCSs over MIMO MAC
- $C_{\text{MAC},\Sigma}^{\text{linfb}}$ = largest sum-rate achieved with LFCSs over MIMO MAC

Linear-Feedback Capacity for SISO MAC

- h_1, h_2: scalars
- Ozarow'84: feedback capacity region under individual power constraints
- sum-power constraint $P \longrightarrow$ union over all $P_1 + P_2 = P$
- Ozarow's scheme is an LMMSE-based linear-feedback scheme

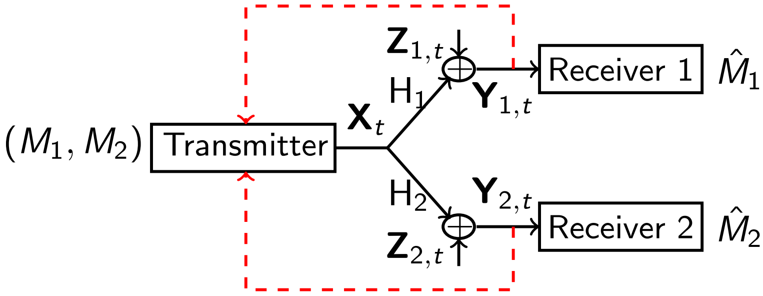
$$\mathcal{C}_{\mathrm{MAC}}^{\mathrm{fb}}(h_1, h_2; P) = \mathcal{C}_{\mathrm{MAC}}^{\mathrm{linfb}}(h_1, h_2; P) =$$

$$\bigcup_{\substack{P_1, P_2 \geq 0: \\ P_1 + P_2 = P}} \bigcup_{\rho \in [0,1]} \left\{ (R_1, R_2) : \begin{array}{rl} R_1 & \leq \frac{1}{2} \log\left(1 + h_1^2 P_1 (1 - \rho^2)\right), \\ R_2 & \leq \frac{1}{2} \log\left(1 + h_2^2 P_2 (1 - \rho^2)\right), \\ R_1 + R_2 & \leq \frac{1}{2} \log\left(1 + h_1^2 P_1 + h_2^2 P_2 + 2\sqrt{h_1^2 h_2^2 P_1 P_2}\rho\right). \end{array} \right\}$$

Linear-Feedback Capacity for SIMO and MISO MAC

- Jafar/Goldsmith'06: feedback capacity region under individual power constraints
- sum-power constraint$\longrightarrow$ union over all $P_1 + P_2 = P$
- a variation of Ozarow's scheme (LMMSE-based LFCS)

2-user MIMO Gaussian BC with Feedback



$$\mathbf{Y}_{i,t} = \mathsf{H}_i \mathbf{x}_t + \mathbf{Z}_{i,t}, \quad i \in \{1,2\}, \quad t \in \{1\ldots,n\}$$

- $\mathsf{H}_i \in \mathbb{R}^{\nu_i \times \kappa}$ deterministic
- $\mathbf{x}_t \in \mathbb{R}^{\kappa}$, $\mathbf{Y}_{i,t} \in \mathbb{R}^{\nu_i}$
- Independent $\{\mathbf{Z}_{1,t}\}_{t=1}^n$ and $\{\mathbf{Z}_{2,t}\}_{t=1}^n$ i.i.d.$\sim \mathcal{N}(0, \mathsf{I})$
- Messages M_1 and M_2 independent ; $M_i \sim \mathcal{U}\{1,\ldots,2^{nR_i}\}$
- Power constraint: $\displaystyle \frac{1}{n}\sum_{t=1}^{n} \mathsf{E}\left[\|\mathbf{X}_t\|^2\right] \leq P$
- Perfect output feedback:

Linear-Feedback Coding Schemes (LFCS) for MIMO BC

- Feedback is used linearly

$$\mathbf{X}_t = \mathbf{V}_t + \sum_{i=1}^{2}\sum_{\tau=1}^{t-1} \mathsf{A}_{i,\tau,t}\mathbf{Y}_{i,\tau}, \quad t \in \{1,\ldots,n\}$$

  - $\mathsf{A}_{i,\tau,t}$ are arbitrary κ-by-ν_i matrices
  - $\mathbf{V}_t \triangleq \xi_t^{(n)}(M_1, M_2), \quad t \in \{1,\ldots,n\}$
- Decoding can be arbitrary
- $\mathcal{C}_{\text{BC}}^{\text{linfb}}$ = all rates achievable with LFCS over MIMO BC
- $C_{\text{BC},\Sigma}^{\text{linfb}}$ = largest sum-rate achievable with LFCS over MIMO BC

Achievable Regions with Feedback for Scalar BC

- Feeback Capacity is open even for scalar BC!!
- Linear-feedback schemes:
  - Ozarow/Leung'84,
  - Kramer'02,
  - Gastpar/Lapidoth/Steinberg/Wigger'11,
  - Ardestanizadeh/Minero/Franceschetti'12 ...
- Non-linear feedback schemes:
  - Shayewitz/Wigger'13,
  - Venkataramanan/Pradhan'11,
  - Wu/Wigger'14 (rate-limited feedback → see tomorrow's talk!)

- Best achievable regions by LFCS

Previous Duality Results

- (Ardestanizadeh, Minero, Franceschetti'12) for scalar BC:
  - based on LQG control theory (not on LMMSEs!)
  - achieves MAC-sum-capacity when $h_1 = h_2$ (over BC) (duality !)

No Feedback Duality:
(Vishwanath/Jindal/Goldsmith'03, Viswanath/Tse'03, Weingarten/Steinberg/Shamai'06)

$$\mathcal{C}_{\text{MAC}}^{\text{nofb}}(\mathsf{H}_1^{\mathsf{T}}, \mathsf{H}_2^{\mathsf{T}}; P) = \mathcal{C}_{\text{BC}}^{\text{nofb}}(\mathsf{H}_1, \mathsf{H}_2; P)$$

Main Results

Multi-Antenna MAC-BC Duality with Linear-Feedback Schemes

$$\mathcal{C}^{\text{linfb}}_{\text{MAC}}(\mathsf{H}_1^{\mathsf{T}}, \mathsf{H}_2^{\mathsf{T}}; P) = \mathcal{C}^{\text{linfb}}_{\text{BC}}(\mathsf{H}_1, \mathsf{H}_2; P)$$

Corollary on Linear-feedback Sum-Capacities

$$C^{\text{linfb}}_{\text{MAC},\Sigma}(\mathsf{H}_1^{\mathsf{T}}, \mathsf{H}_2^{\mathsf{T}}; P) = C^{\text{linfb}}_{\text{BC},\Sigma}(\mathsf{H}_1, \mathsf{H}_2; P)$$

- ☺ $\mathcal{C}^{\text{linfb}}_{\text{MAC,SISO}}$, $\mathcal{C}^{\text{linfb}}_{\text{MAC,SIMO}}$, and $\mathcal{C}^{\text{linfb}}_{\text{MAC,MISO}}$ are known (Ozarow'84) & (Jafar etal'06)
- ⇒ Obtain $\mathcal{C}^{\text{linfb}}_{\text{BC,SISO}}$, $\mathcal{C}^{\text{linfb}}_{\text{BC,SIMO}}$, and $\mathcal{C}^{\text{linfb}}_{\text{BC,MISO}}$

Main Results

Scalar Gaussian MAC & BC

Scalar Gaussian MAC and BC Duality with Linear-Feedback Schemes

$$\mathcal{C}^{\text{linfb}}_{\text{MAC,SISO}}(h_1, h_2; P) = \mathcal{C}^{\text{linfb}}_{\text{BC,SISO}}(h_1, h_2; P)$$

From Ozarow'84:

$$\mathcal{C}^{\text{linfb}}_{\text{BC,SISO}}(h_1, h_2; P) = \mathcal{C}^{\text{linfb}}_{\text{MAC,SISO}}(h_1, h_2; P) =$$

$$\bigcup_{\substack{P_1, P_2 \geq 0: \\ P_1 + P_2 = P}} \bigcup_{\rho \in [0,1]} \left\{ (R_1, R_2) : \begin{array}{rl} R_1 & \leq \frac{1}{2} \log\left(1 + h_1^2 P_1 (1 - \rho^2)\right), \\ R_2 & \leq \frac{1}{2} \log\left(1 + h_2^2 P_2 (1 - \rho^2)\right), \\ R_1 + R_2 & \leq \frac{1}{2} \log\left(1 + h_1^2 P_1 + h_2^2 P_2 + 2\sqrt{h_1^2 h_2^2 P_1 P_2 \rho}\right) \end{array} \right\}$$

Main Results

Scalar Gaussian MAC & BC

Corollary on Linear-Feedback Sum-Capacities

$C^{\text{linfb}}_{\text{BC,SISO},\Sigma}(h_1, h_2; P) = C^{\text{linfb}}_{\text{MAC,SISO},\Sigma}(h_1, h_2; P) =$

$$\sup_{\substack{P_1, P_2 \geq 0: \\ P_1 + P_2 = P}} \frac{1}{2} \log \left(1 + h_1^2 P_1 + h_2^2 P_2 + 2\sqrt{h_1^2 h_2^2 P_1 P_2} \cdot \rho^\star(h_1, h_2; P_1, P_2) \right)$$

- $\rho^\star(h_1, h_2; P_1, P_2)$ is the unique solution in $[0, 1]$ to

$$(1 + h_1^2 P_1 + h_2^2 P_2 + 2\sqrt{h_1^2 h_2^2 P_1 P_2 \rho}) = (1 + h_1^2 P_1 (1 - \rho^2))(1 + h_2^2 P_2 (1 - \rho^2))$$

If $h_1 = h_2 = h$:

$$C^{\text{linfb}}_{\text{BC,SISO},\Sigma}(h, h; P) = \frac{1}{2} \log \left(1 + h^2 P + h^2 P \cdot \rho^\star(h, h; P, P) \right)$$

$\Rightarrow$ (Ardestanizadeh, Minero, Franceschetti'12) is optimal

Main Results

Comparison to Other Bounds for the Scalar BC

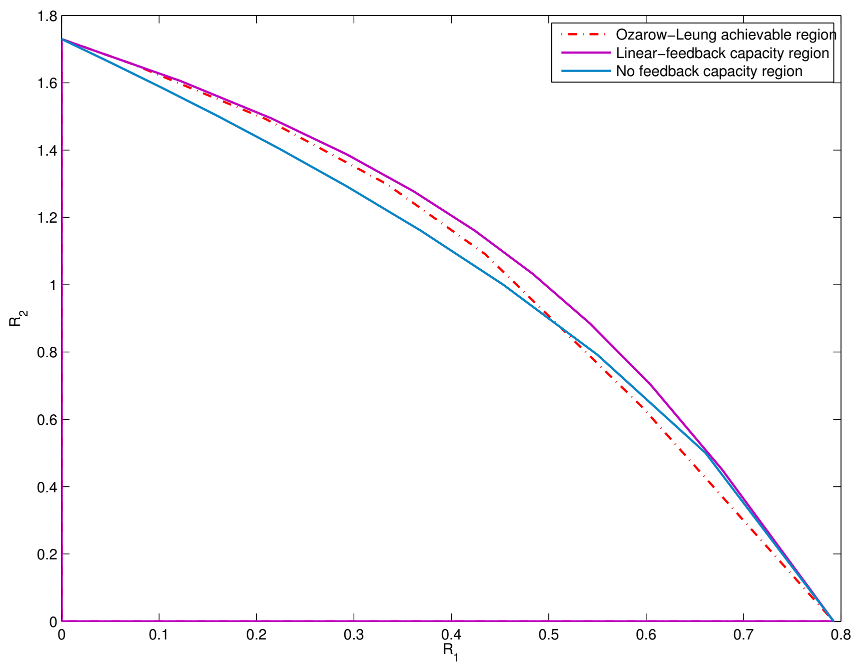


Figure: $h_1 = \frac{1}{\sqrt{5}}$, $h_2 = 1$ and power $P = 10$

Main Results

Linear-feedback Capacity for the MISO Gaussian BC

Linear-feedback Capacity for the MISO Gaussian BC

$$\mathcal{C}^{\text{linfb}}_{\text{BC,MISO}}(\mathbf{h}_1^{\mathsf{T}}, \mathbf{h}_2^{\mathsf{T}}; P) = \mathcal{C}^{\text{linfb}}_{\text{MAC,SIMO}}(\mathbf{h}_1, \mathbf{h}_2; P)$$

From Jafar etal'06:

$$\mathcal{C}^{\text{linfb}}_{\text{BC,MISO}}(\mathbf{h}_1^{\mathsf{T}}, \mathbf{h}_2^{\mathsf{T}}; P) = \mathcal{C}^{\text{linfb}}_{\text{MAC,SIMO}}(\mathbf{h}_1^{\mathsf{T}}, \mathbf{h}_2^{\mathsf{T}}; P) = \mathcal{C}^{\text{fb}}_{\text{MAC,SIMO}}(\mathbf{h}_1^{\mathsf{T}}, \mathbf{h}_2^{\mathsf{T}}; P) =$$

$$\bigcup_{\substack{P_1, P_2 \geq 0: \\ P_1 + P_2 = P}} \bigcup_{\rho \in [0,1]} \left\{ (R_1, R_2) : \begin{array}{rl} R_1 & \leq \frac{1}{2} \log\left(1 + \|\mathbf{h}_1\|^2 P_1 (1-\rho^2)\right), \\ R_2 & \leq \frac{1}{2} \log\left(1 + \|\mathbf{h}_2\|^2 P_2 (1-\rho^2)\right), \\ R_1 + R_2 & \leq \frac{1}{2} \log\left(1 + \|\mathbf{h}_1\|^2 P_1 + \|\mathbf{h}_2\|^2 P_2 + 2\sqrt{\|\mathbf{h}_1\|^2 \|\mathbf{h}_2\|^2 P_1 P_2}\rho\beta \right. \\ & \left. \quad + \|\mathbf{h}_1\|^2 \|\mathbf{h}_2\|^2 P_1 P_2 (1-\rho^2)(1-\beta^2)\right) \end{array} \right\}$$

Main Results

Linear-Feedback Capacity for the SIMO Gaussian BC

Linear-Feedback Capacity for the SIMO Gaussian BC

$$\mathcal{C}^{\text{linfb}}_{\text{BC,SIMO}}(\mathbf{h}_1, \mathbf{h}_2; P) = \mathcal{C}^{\text{linfb}}_{\text{MAC,MISO}}(\mathbf{h}_1^{\mathsf{T}}, \mathbf{h}_2^{\mathsf{T}}; P)$$

From Jafar etal'06:

$$\begin{aligned} \mathcal{C}^{\text{linfb}}_{\text{BC,SIMO}}(\mathbf{h}_1, \mathbf{h}_2; P) &= \mathcal{C}^{\text{linfb}}_{\text{MAC,MISO}}(\mathbf{h}_1^{\mathsf{T}}, \mathbf{h}_2^{\mathsf{T}}; P) \\ &= \mathcal{C}^{\text{fb}}_{\text{MAC,MISO}}(\mathbf{h}_1^{\mathsf{T}}, \mathbf{h}_2^{\mathsf{T}}; P) = \mathcal{C}^{\text{fb}}_{\text{MAC,SISO}}(\|\mathbf{h}_1\|, \|\mathbf{h}_2\|; P) = \end{aligned}$$

$$\bigcup_{\substack{P_1, P_2 \geq 0: \\ P_1 + P_2 = P}} \bigcup_{\rho \in [0,1]} \left\{ (R_1, R_2) : \begin{array}{rl} R_1 & \leq \frac{1}{2} \log\left(1 + \|\mathbf{h}_1\|^2 P_1 (1-\rho^2)\right), \\ R_2 & \leq \frac{1}{2} \log\left(1 + \|\mathbf{h}_2\|^2 P_2 (1-\rho^2)\right), \\ R_1 + R_2 & \leq \frac{1}{2} \log\left(1 + \|\mathbf{h}_1\|^2 P_1 + \|\mathbf{h}_2\|^2 P_2 + 2\sqrt{\|\mathbf{h}_1\|^2 \|\mathbf{h}_2\|^2 P_1 P_2}\rho\right) \end{array} \right\}$$

Sketch of the Proof

1. Optimal LFCSs for MAC and BC

→ Multi-letter expressions for $\mathcal{C}^{\text{linfb}}_{\text{MAC}}(\mathsf{H}_1^{\mathsf{T}}, \mathsf{H}_2^{\mathsf{T}}; P)$ and $\mathcal{C}^{\text{linfb}}_{\text{BC}}(\mathsf{H}_1, \mathsf{H}_2; P)$

2. Identify pairs of MAC/BC params $\longrightarrow$ same performance on MAC and BC

→ For each MAC-params, ∃ BC-params achieving the same region, vice-versa.

Dual Optimal LFCS for MAC and BC

- Divide the blocklength into subblocks of length-η
- Inner Code (described by $\{\mathsf{C}_{i,\tau,\ell}\}$ for MAC and $\{\mathsf{A}_{i,\tau,\ell}\}$ for BC):
  - uses feedback linearly.
  - transforms η channel uses into 1 new super Gaussian MIMO MAC or BC channel use
- Outer Code:
  - ignores feedback
  - codes to achieve nofeedback capacity of super Gaussian MIMO MAC or BC

If we choose

$$\mathsf{A}_{i,\tau,\ell} = \bar{\mathsf{C}}_{i,\eta-\tau,\eta-\ell+2},$$

then super Gaussian MIMO MAC and BC have same nofeedback capacity.

($\bar{\mathsf{C}}$: mirror image of C along counter-diagonal)

- Proof: nofeedback MAC-BC duality & equivalence relations on super Gaussian MIMO channels

⇒ Ozarow'84 & Jafar etal'06 $\longrightarrow$ Optimal BC parameters

Extension I: One-Sided Feedback

- Feedback only from BC-Receiver 1 and only to MAC-Transmitter 1

Multi-Antenna Gaussian MAC-BC Duality with One-sided Linear-Feedback Schemes

$$\mathcal{C}_{\text{MAC}}^{\text{linfb@Tx1}}(\mathsf{H}_1^{\mathsf{T}}, \mathsf{H}_2^{\mathsf{T}}; P) = \mathcal{C}_{\text{BC}}^{\text{linfb@Rx1}}(\mathsf{H}_1, \mathsf{H}_2; P).$$

- $\mathcal{C}_{\text{MAC}}^{\text{linfb@Tx1}}$ and $\mathcal{C}_{\text{BC}}^{\text{linfb@Rx1}}$ are both unknown even in the scalar case
- Lapidoth/Wigger'06 scalar MAC achievable region transfers to scalar BC

  $\longrightarrow$ One-sided feedback always increases capacity of scalar Gaussian BC
- Bhaskaran'08 and Lapidoth/Steinberg/Wigger'10 scalar BC achievable regions transfer to scalar MAC

Extension II: MAC and BC with $K \geq 3$-Users

Multi-Antenna $K \geq 3$-Users MAC-BC Duality with LFCSs

$$\mathcal{C}_{\text{MAC},K}^{\text{linfb}}(\mathsf{H}_1^{\mathsf{T}}, \ldots, \mathsf{H}_K^{\mathsf{T}}; P) = \mathcal{C}_{\text{BC},K}^{\text{linfb}}(\mathsf{H}_1, \ldots, \mathsf{H}_K; P).$$

In the scalar case, if $h_1 = \cdots = h_K = h$,
(Kramer'02, Ardestanizadeh/Wigger/Kim/Javidi'12, a symmetry argument)

$$C_{\text{BC},K,\text{SISO},\Sigma}^{\text{linfb}}(h, \ldots, h; P) = C_{\text{MAC},K,\text{SISO},\Sigma}^{\text{linfb}}(h, \ldots, h; P) = \frac{1}{2}\log\left(1 + P\phi(K,P)\right),$$

where $\phi(K,P)$ is the unique solution in $[1, K]$ to:

$$(1 + P\phi)^{K-1} = \left(1 + \frac{P}{K}\phi(K - \phi)\right).$$

$\Rightarrow$ (Ardestanizadeh, Minero, Franceschetti'12) is optimal for $K \geq 3$-user sym. BC

Summary

Multi-Antenna MAC-BC Duality with Linear-Feedback Schemes

$$\mathcal{C}^{\text{linfb}}_{\text{MAC}}(\mathsf{H}_1^{\mathsf{T}}, \mathsf{H}_2^{\mathsf{T}}; P) = \mathcal{C}^{\text{linfb}}_{\text{BC}}(\mathsf{H}_1, \mathsf{H}_2; P)$$

- Extends to MAC and BC with one-sided feedback
- Extends to MAC and BC with arbitrary $K \geq 2$-users
- Optimal multi-letter LFCS for MAC and BC
- Explicit expressions of $\mathcal{C}^{\text{linfb}}_{\text{BC}}$ and $C^{\text{linfb}}_{\text{BC},\Sigma}$ for SISO, SIMO, and MISO BC
- (Ardestanizadeh/Minero/Franceschetti'12) is sum-rate optimal among LFCS for SISO symmetric BC with $K \geq 2$-users

S. Belhadj Amor, Y. Steinberg, and M. Wigger, "MIMO MAC-BC Duality with Linear-Feedback Coding Schemes," submitted to *IEEE Trans. on Inf. Theory*, March 2014. Posted on *Arxiv*:`http://arxiv.org/abs/1404.2584`.