A New Coding Scheme for Discrete Memoryless MACs with Common Rate-Limited Feedback

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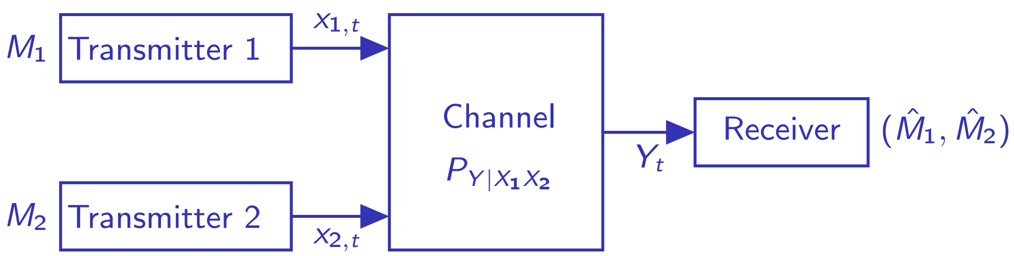
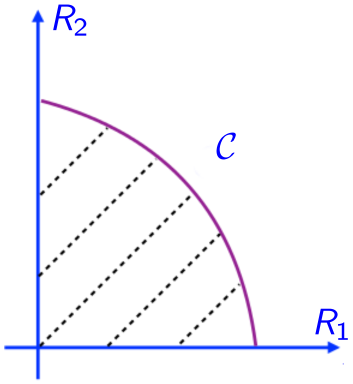
Discrete Memoryless Multi-Access Channel



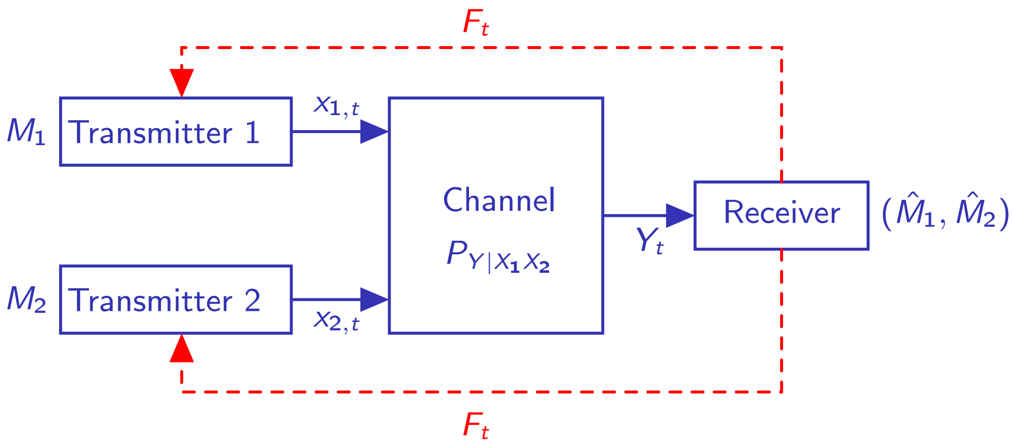
- Finite inputs and output alphabets $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}$
- Memoryless channel $P_{Y|X_1X_2}$
- Messages M_1 and M_2 independent
- $p(error) = \Pr\big((M_1, M_2) \neq (\hat{M}_1, \hat{M}_2)\big)$

Capacity Region

- $M_1 \sim \mathcal{U}\{1, \ldots, \lfloor 2^{nR_1} \rfloor\}$ and $M_2 \sim \mathcal{U}\{1, \ldots, \lfloor 2^{nR_2} \rfloor\}$
- n: blocklength
- $R_1, R_2 \geq 0$: rates of communications
- Capacity-region $\mathcal{C}$: Pairs (R_1, R_2) s.t. $p(error)$ can be made arbitrarily small.



Rate-Limited Feedback Pipes ($R_{fb} \geq 0$)



- Feedback signals: $F_t = \psi_t^{(n)}(Y_1, \ldots, Y_t) \in \mathcal{F}_t$.
- Both transmitters receive feedback signals perfectly whenever

$$|\mathcal{F}_1| \times \ldots \times |\mathcal{F}_n| \leq 2^{nR_{fb}}.$$

- Inputs: $X_{i,t} = \varphi_{i,t}^{(n)}(M_i, F_1, \ldots, F_{t-1})$

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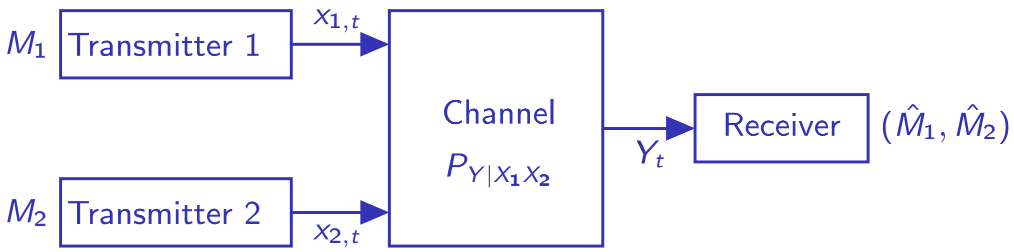

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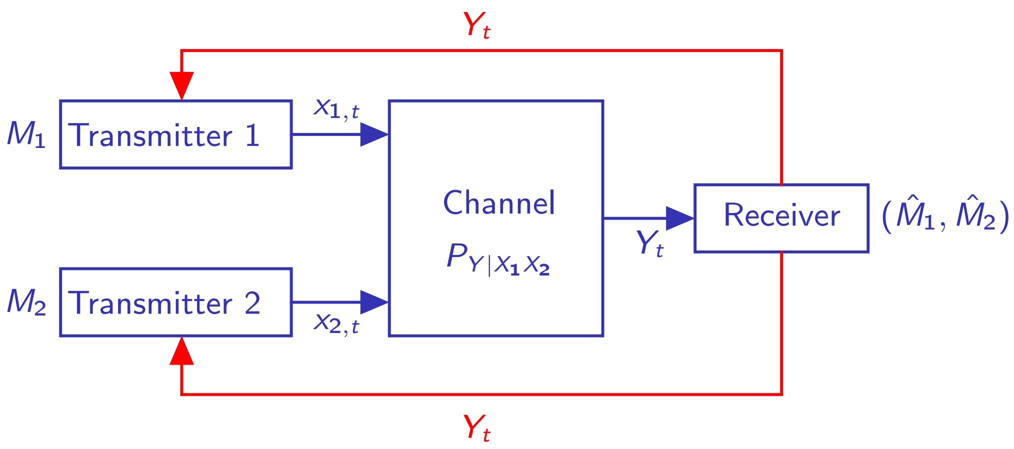

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- Inputs: $X_{i,t} = \varphi_{i,t}^{(n)}(M_i, F_1, \ldots, F_{t-1})$
- Nofeedback: $R_{\text{fb}} = 0$
- Perfect feedback: $R_{\text{fb}} \geq \log_2 |\mathcal{Y}|$

Nofeedback Capacity Region ($R_{fb} = 0$)

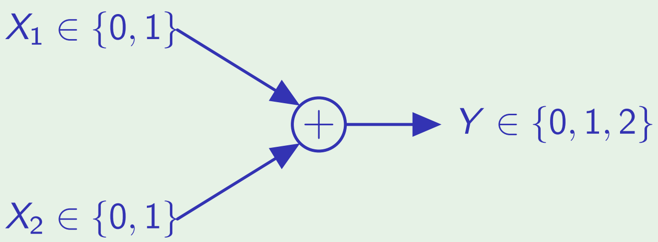
Nofeedback capacity region $\mathcal{C}_{nofb}$ (Ahlswede'71 and Liao'72)

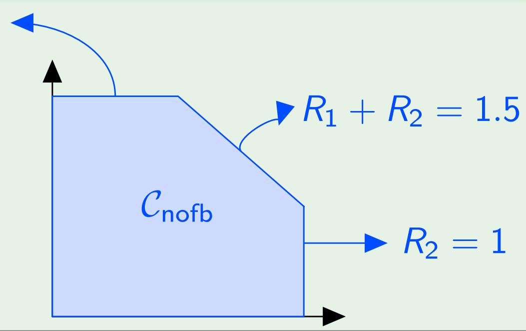
All nonnegative rate-pairs (R_1, R_2):

$$
\begin{aligned}
R_1 &\le I(X_1; Y|X_2), \\
R_2 &\le I(X_2; Y|X_1), \\
R_1 + R_2 &\le I(X_1X_2; Y),
\end{aligned}
$$

for some $(X_1, X_2) \sim P_{X_1}P_{X_2}$.

Example





Feedback Increases Capacity – Gaarder & Wolf 1975

	Information phase	Update phase
Tx 1	1 1 0 1 1 0 0 0 1 0 1 1 1 0 0 0 0 0 1 1	
Tx 2	0 1 1 0 1 0 0 1 0 1 1 0 1 0 1 0 1 0 0 1	
Rx	1 2 1 1 2 0 0 1 1 1 2 1 2 0 1 0 1 0 1 2	

- Each Tx sends K uncoded bits

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- Each Tx sends K uncoded bits
- Outputs 0 and 2 perfectly decoded; Output 1: $(1,0)$ or $(0,1)$ $\rightarrow$ 'collision'

$\rightarrow$ On average $\frac{K}{2}$ collisions!

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- Perfect feedback: each Tx knows $Y_1, \ldots, Y_K$ $\Rightarrow$ locations of collisions

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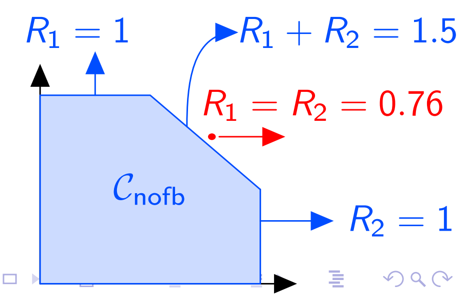
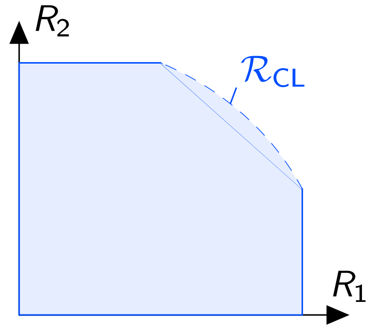
- Txs cooperate and use 3 input symbols over a noiseless channel at rate $\log_2 3$



Cover-Leung Achievable Region $\mathcal{R}_{CL}$

- Cover & Leung 1981 extended G-W scheme

→ Interleaving 2 phases of G-W scheme



Cover-Leung Achievable Region $\mathcal{R}_{CL}$

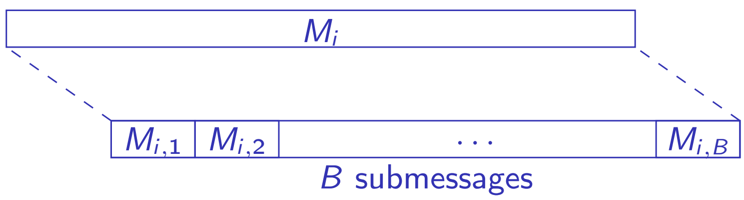
All nonnegative rate-pairs (R_1, R_2) satisfying

$$\begin{aligned} R_1 &\leq I(X_1; Y|X_2 W), \\ R_2 &\leq I(X_2; Y|X_1 W), \\ R_1 + R_2 &\leq I(X_1 X_2; Y), \end{aligned}$$

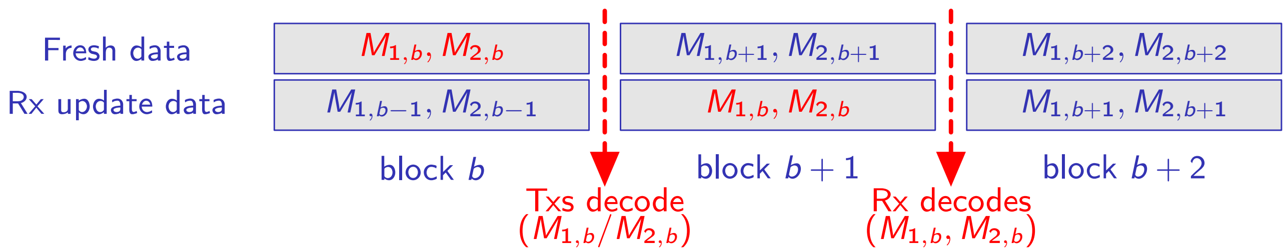
for some $P_W P_{X_1|W} P_{X_2|W}$ s.t. $|\mathcal{W}| \leq \min\{|\mathcal{X}_1||\mathcal{X}_2| + 1, |\mathcal{Y}| + 2\}$.

Cover-Leung Coding Scheme

- Each message M_i is split into B submessages:



- Block-Markov coding over $B+1$ blocks
- In Block b, send fresh data $(M_{1,b}, M_{2,b})$ superposed on update info about $(M_{1,b-1}, M_{2,b-1})$.



- After transmission of Block b:
  - Tx 1 (2) decodes $M_{2,b}$ $(M_{1,b})$
  - Rx decodes $(M_{1,b-1}, M_{2,b-1})$

Cover-Leung Coding Scheme

- Willems'82 converse : C-L scheme optimal for MACs such that

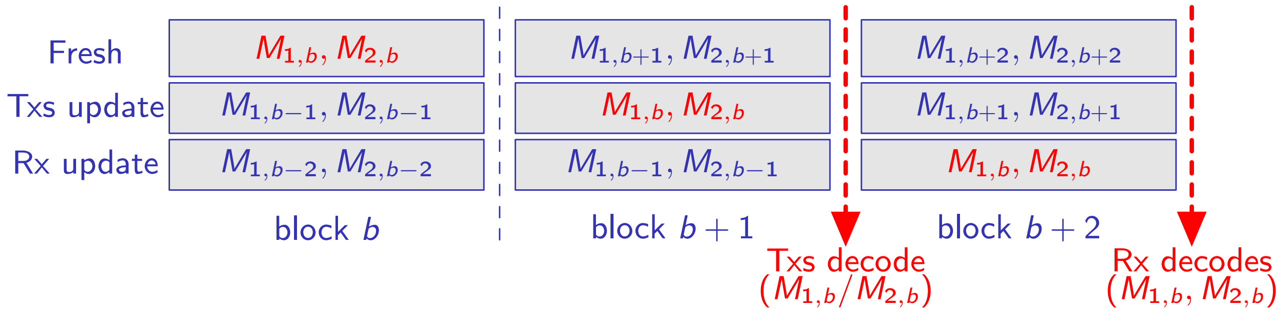
$$X_1 = f(X_2, Y) \quad \text{or} \quad X_2 = f(X_1, Y).$$

- Not optimal in general (Gaussian MAC)
- Improved by delaying the decoding at Txs (and at Rx)

$\Rightarrow$ Txs gain more info about each other's message.

Venkataraman-Pradhan Coding Scheme

- Venkataramanan & Pradhan (2009)
  - delay Txs decoding of the fresh data by an entire block.
  - 2 sorts of update info in Block b,
    - ⋆ common Rx-side update info about Block $b-2$
    - ⋆ correlated Txs-side update info about Block $b-1$



  - $\mathcal{R}_{\text{VP}} \supset \mathcal{R}_{\text{CL}}$

Previous Results with Common Rate-Limited Feedback

- Shaviv & Steinberg (2008)
  - Carleial's extension of C-L scheme is used over the "forward channel"
    - ★ If in C-L scheme feedback is 'imperfect' $\rightarrow$ Txs can still understand useful update info; Txs can decode each other message or some part of it
  - Heegard-Berger source-coding on the feedback links

- No feedback $\Rightarrow \mathcal{R}_{\text{SS}} = \mathcal{C}_{\text{nofb}}$
- Perfect feedback $\Rightarrow \mathcal{R}_{\text{SS}} = \mathcal{R}_{\text{CL}}$

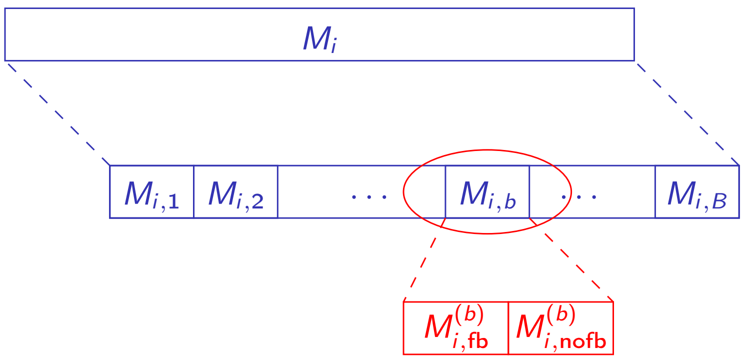
Can we do better?

New Coding Scheme with Common Rate-Limited Feedback

Main Idea

- In S-S scheme, replace Carleial/C-L scheme by V-P scheme

- Each message M_i is split into B submessages
- Each submessage $M_i^{(b)}$ is split into $M_{i,\text{fb}}^{(b)}$ and a $M_{i,\text{nofb}}^{(b)}$

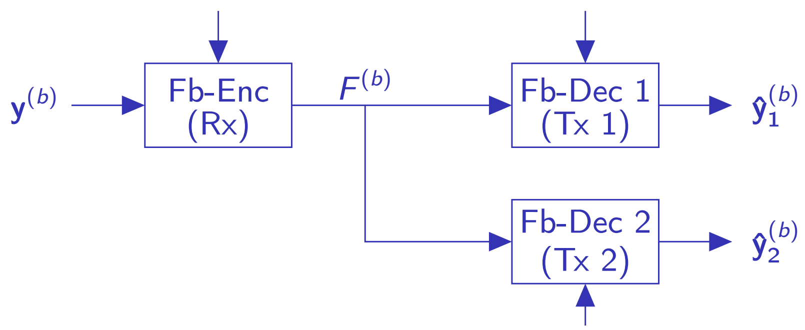


New Coding Scheme with Common Rate-Limited Feedback

- Superposition & block-Markov coding over $B+2$ blocks
- $M_{i,\text{fb}}^{(b)}$ is sent using a feedback code (V-P perfect feedback scheme)
- $M_{i,\text{nofb}}^{(b)}$ is sent using a nofeedback scheme
  1. $(M_{1,\text{fb}}^{(b)}, M_{2,\text{fb}}^{(b)})$ are sent through (U_1, U_2)-codewords in block b
  2. Txs send Tx-side update info through (V_1, V_2)-codewords in block $b+1$
     - ⋆ correlated info can be sent more efficiently than indep. info
     - ⋆ conditions on $P_{V_1V_2}$ to ensure stationarity
  3. After block $b+1$, Txs decode $(M_{1,\text{fb}}^{(b)}, M_{2,\text{fb}}^{(b)})$ using feedback
  4. Txs jointly send common Rx-side update info about $(M_{1,\text{fb}}^{(b)}, M_{2,\text{fb}}^{(b)})$ through W-codewords in block $b+2$
  5. After block $b+2$, Rx decodes $(M_{1,\text{fb}}^{(b)}, M_{2,\text{fb}}^{(b)})$, then decodes $(M_{1,\text{nofb}}^{(b)}, M_{2,\text{nofb}}^{(b)})$.

New Coding Scheme with Common Rate-Limited Feedback

- Feedback links are used at the end of each block to send compression information about channel outputs
- Feedback encoding = Heegard-Berger'85 source-coding:



Achievable Region

Let $(W, V_1, V_2, U_1, U_2, X_1, X_2, Y, Y_{12}, Y_1, Y_2)$ and $(\tilde{W}, \tilde{V}_1, \tilde{V}_2, \tilde{U}_1, \tilde{U}_2, \tilde{X}_1, \tilde{X}_2, \tilde{Y}, \tilde{Y}_{12}, \tilde{Y}_1, \tilde{Y}_2)$ be 2 correlated random tuples s.t.:

1. Joint distributions of the tuples coincide:
$$P_{WU_1U_2V_1V_2X_1X_2YY_{12}Y_1Y_2} = P_{\tilde{W}\tilde{U}_1\tilde{U}_2\tilde{V}_1\tilde{V}_2\tilde{X}_1\tilde{X}_2\tilde{Y}\tilde{Y}_{12}\tilde{Y}_1\tilde{Y}_2},$$
2. Each factors:
$$\begin{aligned} &P_{WU_1U_2V_1V_2X_1X_2YY_{12}Y_1Y_2} \\ &\quad = P_W P_{V_1V_2} P_{U_1|WV_1} P_{U_2|WV_2} P_{X_1|WU_1V_1} P_{X_2|WU_2V_2} P_{Y|X_1X_2} \\ &\qquad \cdot P_{Y_{12}|YW} P_{Y_2|WYY_{12}} P_{Y_1|WYY_{12}} \end{aligned}$$
3. Defining $\tilde{S} \triangleq (\tilde{W}, \tilde{V}_1, \tilde{V}_2, \tilde{Y}_{12})$, the joint distribution over both tuples factors:
$$\begin{aligned} &P_{WU_1U_2V_1V_2X_1X_2YY_{12}Y_1Y_2\tilde{S}\tilde{U}_1\tilde{U}_2\tilde{X}_1\tilde{X}_2\tilde{Y}\tilde{Y}_1\tilde{Y}_2} \\ &\quad = P_{\tilde{S}\tilde{U}_1\tilde{U}_2\tilde{X}_1\tilde{X}_2\tilde{Y}\tilde{Y}_1\tilde{Y}_2} P_{V_1|\tilde{S}\tilde{U}_1} P_{V_2|\tilde{S}\tilde{U}_2} P_{WU_1U_2X_1X_2Y_{12}Y_1Y_2Y|V_1V_2}. \end{aligned}$$

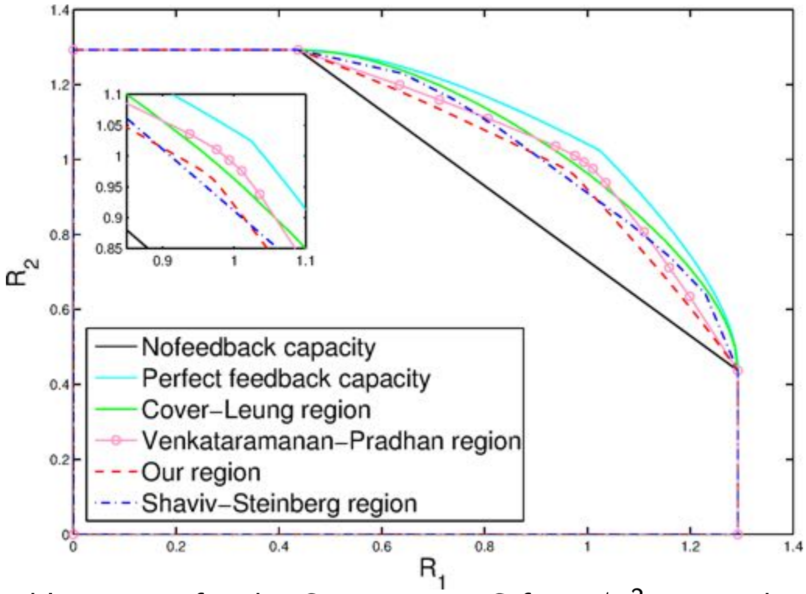
Achievable Region

Achievable Region $\mathcal{R}_{\text{NEW}}$

$$\begin{aligned} R_{\text{fb}} &> \max\{I(Y_{12}; Y|WX_1), I(Y_{12}; Y|WX_2)\} + \sum_{i\in\{1,2\}} I(Y; Y_i|Y_{12}X_iW), \\ R_1 - R_1' &< I(X_1; Y|SU_1U_2X_2) \\ R_2 - R_2' &< I(X_2; Y|SU_1U_2X_1) \\ R_1 - R_1' + R_2 - R_2' &< I(X_1X_2; Y|SU_1U_2) \\ R_1' &< I(U_1; Y_2Y_{12}|\tilde{S}\tilde{Y}_2\tilde{U}_2\tilde{X}_2WV_2U_2X_2) \\ R_2' &< I(U_2; Y_1Y_{12}|\tilde{S}\tilde{Y}_1\tilde{U}_1\tilde{X}_1WV_1U_1X_1) \\ R_1' &< I(W; Y|\tilde{W}\tilde{Y}) + I(U_1; Y|WV_1V_2U_2) + I(V_1; Y|\tilde{Y}\tilde{S}\tilde{Y}_1\tilde{Y}_2\tilde{U}_2WV_2) \\ R_2' &< I(W; Y|\tilde{W}\tilde{Y}) + I(U_2; Y|WV_1V_2U_1) + I(V_2; Y|\tilde{Y}\tilde{S}\tilde{Y}_1\tilde{Y}_2\tilde{U}_1WV_1) \\ R_1' + R_2' &< I(W; Y|\tilde{W}\tilde{Y}) + I(U_1U_2; Y|WV_1V_2) + I(V_1V_2; Y|\tilde{Y}\tilde{S}\tilde{Y}_1\tilde{Y}_2W). \end{aligned}$$

Achievable Region

- No feedback: $\mathcal{R}_{\text{NEW}} = \mathcal{C}_{\text{nofb}}$
- $\mathcal{R}_{\text{NEW}} \supseteq \mathcal{R}_{\text{SS}}$ and inclusion can be strict:
  - Perfect feedback: $\mathcal{R}_{\text{NEW}} = \mathcal{R}_{\text{VP}} \supset \mathcal{R}_{\text{CL}} = \mathcal{R}_{\text{SS}}$
  - Specific (Gaussian) choice of RVs for 2-user Gaussian MAC $\rightarrow$ New scheme achieves a strictly larger sumrate than S-S scheme



Achievable regions for the Gaussian MAC for $P/\sigma^2 = 5$ and $R_{\text{fb}} = 2$