Linear-Feedback MAC-BC Duality for Correlated BC-Noises, and Iterative Coding

Selma Belhadj Amor
{selma.belhadj-amor@inria.fr}

Based on joint work with Michèle Wigger
{michele.wigger@telecom-paristech.fr}

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Contributions

- BC-noises correlation factor $\lambda \in [-1, 1]$
- MAC with "non-standard" sum-power constraint depending on λ

MAC-BC Duality with Correlated Noises at BC-Receivers

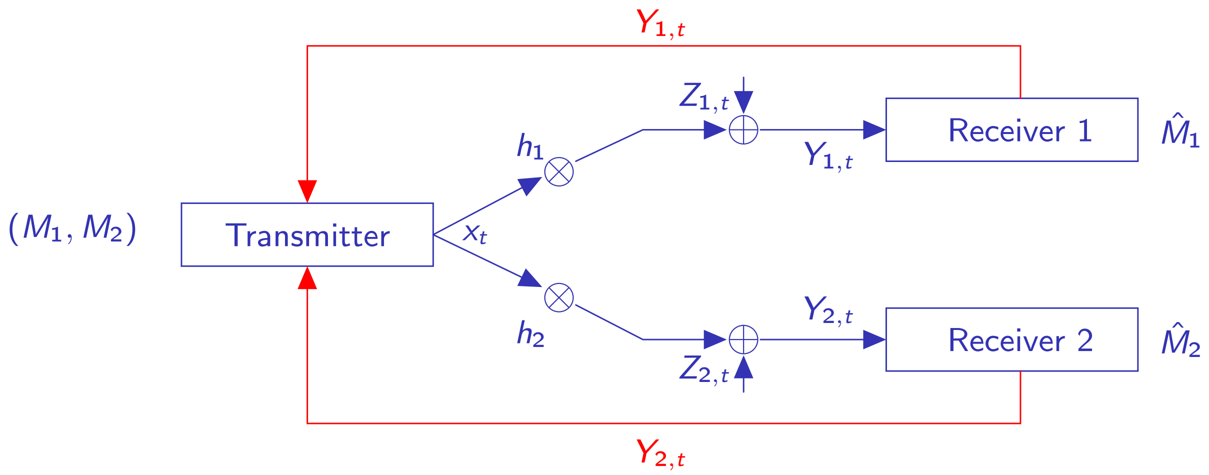
$$\mathcal{C}_{\text{BC}}^{\text{linfb}}(h_1, h_2, \lambda; P) = \mathcal{C}_{\text{BC}}^{\text{linfb}}(h_1, h_2; \lambda, P).$$

- Transfer MAC results to BC
  - (Ozarow'84) $\Longrightarrow$ Achievable region for BC with correlated noises

Constructive Sum-Rate Optimal BC-Scheme ($\lambda = 0$)

- Ozarow's MAC-encoders and MAC-decoders "rearranged" $\Rightarrow$ Constructive sum-rate optimal BC-scheme.

Two-User Gaussian BC with Perfect Feedback



- Transmitter sends M_1 to Receiver 1 and M_2 to Receiver 2.
- Messages M_1 and M_2 independent ; $M_i \sim \mathcal{U}\{1, \ldots, 2^{nR_i}\}$
- $\Pr(error) = \Pr\left\{(\hat{M}_1 \neq M_1) \text{ or } (\hat{M}_2 \neq M_2)\right\}$

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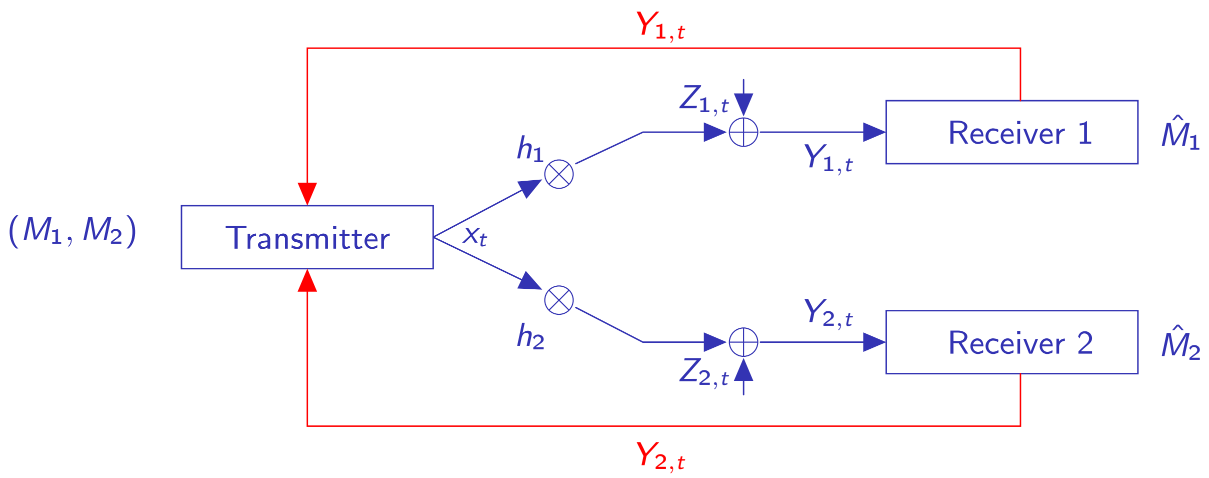
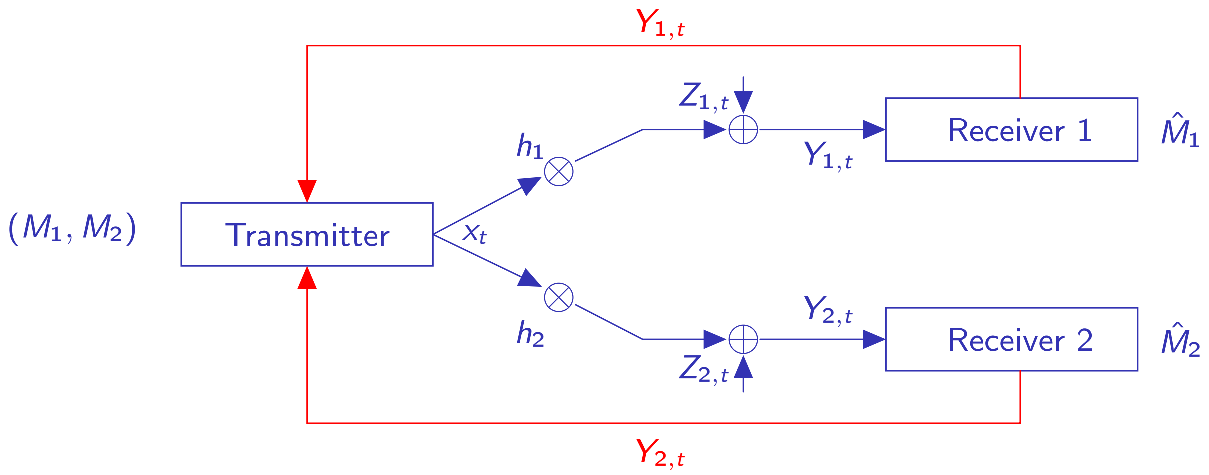


- $h_1, h_2 \in \mathbb{R}$ deterministic, non-fading; $X_t, Y_{1,t}, Y_{2,t} \in \mathbb{R}$
- $Y_{i,t} = h_i x_t + Z_{i,t}$,
- $\begin{pmatrix} Z_{1,t} \\ Z_{2,t} \end{pmatrix}$ i.i.d. $\sim \mathcal{N}\left(\mathbf{0}, \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix}\right)$ where $\lambda \in [-1, 1]$
- BC is physically-degraded if: $\lambda = \frac{h_1}{h_2}$ or $\lambda = \frac{h_2}{h_1}$.

Two-User Gaussian BC with Perfect Feedback



- Power constraint: $\mathsf{E}\left[\|\mathbf{X}^n\|^2\right] \leq nP$; $\qquad \mathbf{X}^n \triangleq (X_1, \ldots, X_n)^\mathsf{T}$
- Perfect output feedback:

$$X_t = \varphi_t^{(n)}(M_1, M_2, Y_{1,1}, \ldots, Y_{1,t-1}, Y_{2,1}, \ldots, Y_{2,t-1}), \quad t \in \{1, \ldots, n\}.$$

Capacity of Gaussian BC with Feedback

- Capacity region $\mathcal{C}_{\mathrm{BC}}^{\mathrm{fb}}$ unknown
- Sum-capacity $C_{\mathrm{BC},\Sigma}^{\mathrm{fb}}$ at high SNR: $(\lim_{P\to\infty} C_{\mathrm{BC},\Sigma}^{\mathrm{fb}} - C_{\mathrm{High}} = 0)$
  (Gastpar *et al.*'14)

$$C_{\mathrm{High}} = \begin{cases} \frac{1}{2}\log\left(1 + \frac{h_1^2 + h_2^2 - 2h_1h_2\lambda}{1-\lambda^2}P\right) & \text{if } -1 < \lambda < 1, \\ \frac{1}{2}\log\left(1 + h_1^2P\right) + \frac{1}{2}\log\left(1 + h_2^2P\right) & \text{if } \lambda \in \{-1, 1\} \text{ and } h_1 \neq \lambda h_2 \\ \frac{1}{2}\log\left(1 + h_1^2P\right) & \text{if } \lambda \in \{-1, 1\} \text{ and } h_1 = \lambda h_2. \end{cases}$$

Achievable Regions with Feedback for Gaussian BC

- Ozarow/Leung'84, Kramer'02: LMMSE-based Schalwijk-Kailath-type schemes
- Elia'04, Wu *et al.*'05, Ardestanizadeh *et al.*'12: Control-theory based schemes
- Gastpar *et al.*'11,

Linear-Feedback Schemes

- Venkataramanan/Pradhan'11, Shayevitz/Wigger'13, Wu/Wigger'14

Non-Linear-Feedback Schemes

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Linear-Feedback Schemes $\rightarrow$ best regions!

- Venkataramanan/Pradhan'11, Shayevitz/Wigger'13, Wu/Wigger'14

Non-Linear-Feedback Schemes

Linear-Feedback Schemes (LFSs) for BC

- Feedback used linearly:

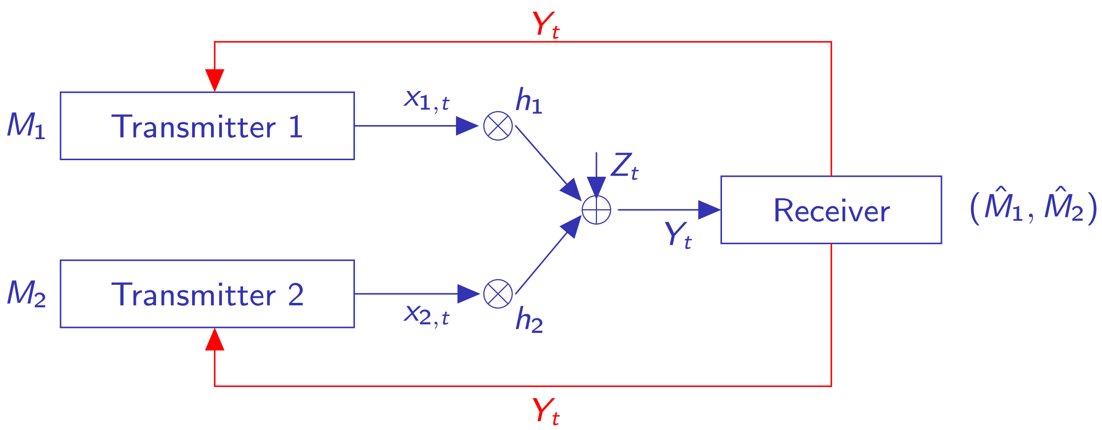
$$\mathbf{X}^n = \mathbf{W}^n + \mathsf{A}_{1,\text{BC}}\mathbf{Y}_1^n + \mathsf{A}_{2,\text{BC}}\mathbf{Y}_2^n,$$

$\mathsf{A}_{1,\text{BC}}, \mathsf{A}_{2,\text{BC}}$: strictly lower-triangular matrices,
$\mathbf{X}^n \triangleq (X_1 \dots X_n)^\mathsf{T}, \quad \mathbf{Y}_i^n \triangleq (Y_{i,1} \dots Y_{i,n})^\mathsf{T}, \quad \mathbf{W}^n \triangleq \xi^{(n)}(M_1, M_2) \in \mathbb{R}^n.$

- Decoding can be arbitrary

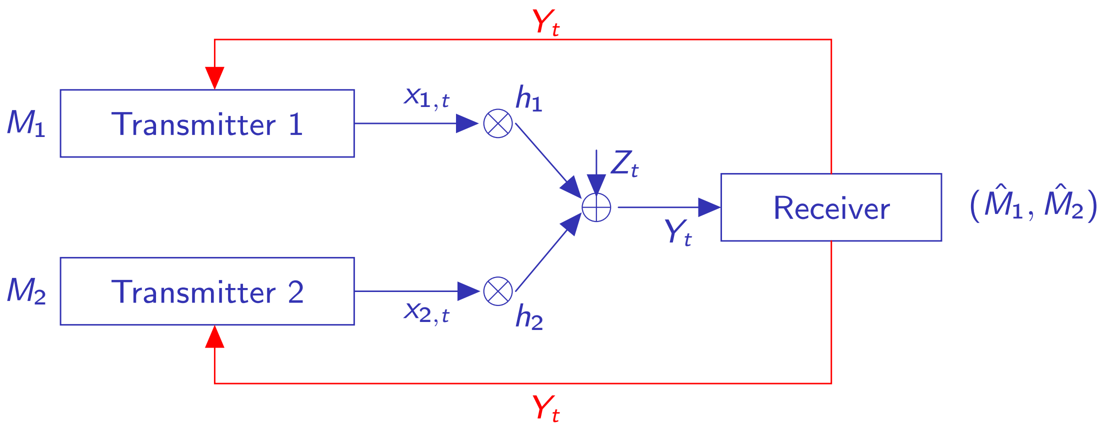
- $\mathcal{C}_{\text{BC}}^{\text{linfb}}(h_1, h_2, \lambda; P)$ = all rates achievable with LFCS over MIMO BC
  - ☹ Optimization step to determine $\mathcal{C}_{\text{BC}}^{\text{linfb}}$ is very difficult.
  - → Tricky part: Identify optimal feedback matrices $\mathsf{A}_{1,\text{BC}}^\star$ and $\mathsf{A}_{2,\text{BC}}^\star$.

Two-User Memoryless Gaussian MAC with Feedback



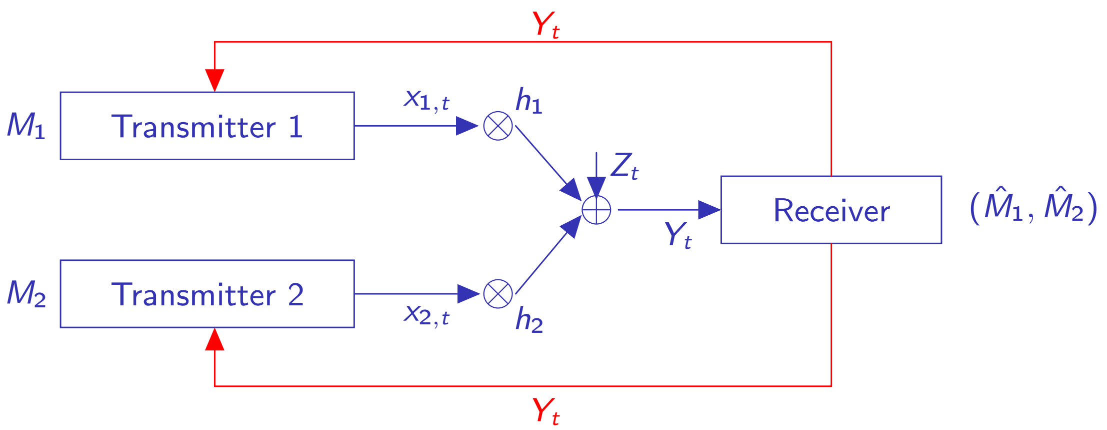
- Transmitters 1 and 2 send independent messages M_1 and M_2 to Receiver
- $M_i \sim \mathcal{U}\{1, \dots, 2^{nR_i}\}$,
- $\Pr(error) = \Pr\Big\{(\hat{M}_1, \hat{M}_2) \neq (M_1, M_2)\Big\}$

Two-User Memoryless Gaussian MAC with Feedback



- $h_i \in \mathbb{R}$: deterministic non-fading,
- $X_{1,t}, X_{2,t}, Y_t \in \mathbb{R}$,
- $Y_t = h_1 X_{1,t} + h_2 X_{2,t} + Z_t$,
- $\{Z_t\}$ i.i.d.$\sim \mathcal{N}(0, 1)$.

Two-User Memoryless Gaussian MAC with Feedback



- $\mathbf{X}_i^n \triangleq (X_{i,1} \dots X_{i,n})^\mathsf{T}, \quad i \in \{1, 2\}$.
- Sum-power constraint:

$$\mathsf{E}\left[\|\mathbf{X}_1^n\|^2\right] + \mathsf{E}\left[\|\mathbf{X}_2^n\|^2\right] \leq nP.$$

- Perfect output feedback:

$$X_{i,t} = \varphi_{i,t}^{(n)}(M_i, Y_1, \dots, Y_{t-1}), \quad i \in \{1, 2\}, \quad t \in \{1, \dots, n\}.$$

Linear-Feedback Schemes (LFSs) for MAC

- Feedback used linearly:

$$\mathbf{X}_i^n = \mathbf{V}_i^n + \mathsf{A}_{i,\mathsf{MAC}} \mathbf{Y}^n, \quad i \in \{1,2\},$$

$\mathsf{A}_{1,\mathsf{MAC}}, \mathsf{A}_{2,\mathsf{MAC}}$: strictly lower-triangular matrices,
$\mathbf{X}_i^n \triangleq (X_{i,1} \dots X_{i,n})^\mathsf{T}, \quad \mathbf{Y}^n \triangleq (Y_1 \dots Y_n)^\mathsf{T}, \quad \mathbf{V}_i^n \triangleq \varphi_i^{(n)}(M_i) \in \mathbb{R}^n.$

- $\mathcal{C}_{\mathsf{MAC}}^{\mathsf{linfb}}(h_1, h_2; P) =$ all rate pairs achievable with LFSs over MAC.

Linear-Feedback Capacity for MAC

- Ozarow'84: feedback capacity region under individual power constraints
- sum-power constraint $P \longrightarrow$ union over all $P_1 + P_2 = P$
- LMMSE-based LFS that achieves (linear-) feedback capacity:

$$\mathcal{C}_{\mathsf{MAC}}^{\mathsf{fb}}(h_1, h_2; P) = \mathcal{C}_{\mathsf{MAC}}^{\mathsf{linfb}}(h_1, h_2; P) =$$

$$\bigcup_{\substack{P_1, P_2 \geq 0: \\ P_1 + P_2 = P}} \bigcup_{\rho \in [0,1]} \left\{ (R_1, R_2) : \begin{array}{rl} R_1 & \leq \frac{1}{2} \log\left(1 + h_1^2 P_1 (1 - \rho^2)\right), \\ R_2 & \leq \frac{1}{2} \log\left(1 + h_2^2 P_2 (1 - \rho^2)\right), \\ R_1 + R_2 & \leq \frac{1}{2} \log\left(1 + h_1^2 P_1 + h_2^2 P_2 + 2\sqrt{h_1^2 h_2^2 P_1 P_2} \rho\right). \end{array} \right\}$$

Previous MAC-BC Duality: Independent BC-Noises ($\lambda = 0$)

MAC-BC Duality with Linear-Feedback Schemes—Independent BC-noises
Belhadj Amor/Steinberg/Wigger'14

$$\mathcal{C}_{\mathrm{BC}}^{\mathrm{linfb}}(h_1, h_2, 0; P) = \mathcal{C}_{\mathrm{MAC}}^{\mathrm{linfb}}(h_1, h_2; P) = \mathcal{C}_{\mathrm{MAC}}^{\mathrm{fb}}(h_1, h_2; P).$$

- Explicit expression of $\mathcal{C}_{\mathrm{BC}}^{\mathrm{linfb}}(h_1, h_2, 0; P)$.
- Feeback always increases capacity of Gaussian BC with independent noises.

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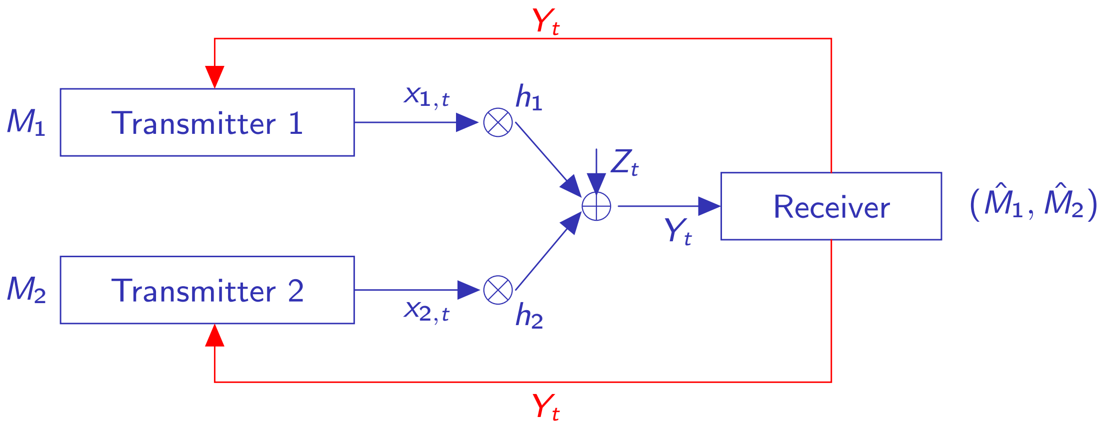
- Explicit expression of $\mathcal{C}_{\mathrm{BC}}^{\mathrm{linfb}}(h_1, h_2, 0; P)$.
- Feeback always increases capacity of Gaussian BC with independent noises.

What happens if the BC-noises are correlated ($\lambda \neq 0$)?

Dual MAC with "Non-Standard" Power Constraint



- $\lambda \in [-1, 1]$: BC-noise correlation factor
- "Non-standard" sum-power constraint:

$$\mathsf{E}\left[\|\mathbf{X}_1^n\|^2\right] + \mathsf{E}\left[\|\mathbf{X}_2^n\|^2\right] + 2\lambda \mathsf{E}[\langle \mathbf{X}_1^n, \mathbf{X}_2^n \rangle] \leq nP \quad (1)$$

- $\mathcal{C}^{\text{linfb}}_{\text{MAC,CorrPower}}(h_1, h_2; \lambda, P)$: all rates achievable with LFSs over this MAC.

Main Results

MAC-BC Duality for BC with Correlated Noises

$$\mathcal{C}^{\text{linfb}}_{\text{BC}}(h_1, h_2, \lambda; P) = \mathcal{C}^{\text{linfb}}_{\text{MAC,CorrPower}}(h_1, h_2; \lambda, P).$$

Main Results

- Let $|h_1| \geq |h_2|$.
- Assume BC is physically degraded, i.e. $\lambda = \frac{h_2}{h_1}$.
- MAC power constraint:

$$\underbrace{\mathrm{E}\left[\|h_1\mathbf{X}_1^n + h_2\mathbf{X}_2^n\|^2\right]}_{\text{received power}} + (h_1^2 - h_2^2)\mathrm{E}\left[\|\mathbf{X}_2^n\|^2\right] \leq h_1^2 nP$$

- Since $(h_1^2 - h_2^2)\mathrm{E}\left[\|\mathbf{X}_2^n\|^2\right] \geq 0 \quad \Rightarrow \quad \underbrace{\mathrm{E}\left[\|h_1\mathbf{X}_1^n + h_2\mathbf{X}_2^n\|^2\right]}_{\text{received power}} \leq h_1^2 nP$.
- With and without feedback

$$C_{\mathrm{MAC},\Sigma} \leq \frac{1}{2}\log\left(1 + (\frac{1}{n})\mathrm{E}\left[\|h_1\mathbf{X}_1 + h_2\mathbf{X}_2\|^2\right]\right) \leq \frac{1}{2}\log(1 + h_1^2 P)$$

- Gastpar'04: Feedback does not increase $C_{\mathrm{MAC},\Sigma}$ under consideration.

→ Feedback does not increase sum-capacity of dual physically-degraded BC.

Main Results

- Adapt Ozarow'84's scheme under ind. powers P_1 and P_2 to our dual MAC:
  - Choose $P_1 \geq 0$ and $P_2 \geq 0$ so that

$$P_1 + P_2 + 2\lambda\rho\sqrt{P_1 P_2} \leq P,$$

  for some parameter $\rho \in [0, \rho^\star(h_1, h_2, P_1, P_2)]$.

- $\rho^\star(h_1, h_2, P_1, P_2)$ is the unique solution in $(0, 1)$ to:

$$(1+h_1^2P_1(1-x^2))(1+h_2^2P_2(1-x^2)) = 1+h_1^2P_1+h_2^2P_2+2|h_1||h_2|x\sqrt{P_1P_2}.$$

Main Results

New Achievable Region over BC with Feedback and Correlated Noises

$$R_1 \le \frac{1}{2}\log(1 + h_1^2 P_1(1-\rho^2))$$
$$R_2 \le \frac{1}{2}\log(1 + h_1^2 P_2(1-\rho^2))$$
$$R_1 + R_2 \le \frac{1}{2}\log(1 + h_1^2 P_1 + h_2^2 P_2 + 2|h_1||h_2|\rho\sqrt{P_1 P_2})$$

for some P_1, P_2, and $\rho \in [0,1]$ satisfying

$$P_1 + P_2 + 2\lambda\rho\sqrt{P_1 P_2} \le P,$$

for some parameter $\rho \in [0, \rho^\star(h_1, h_2, P_1, P_2)]$.

Proof–Step 1: Optimal Block-Feedback Schemes

- Class of MAC- and BC-block-feedback schemes is optimal
  - Divide the blocklength into subblocks of length-η
  - Inner Code (described by $\{\mathsf{A}_{1,\mathrm{MAC}}, \mathsf{A}_{2,\mathrm{MAC}}\}$ for MAC and $\{\mathsf{A}_{1,\mathrm{BC}}, \mathsf{A}_{2,\mathrm{BC}}\}$ for BC):
    - uses feedback linearly.
    - transforms η channel uses into 1 channel use of new super Gaussian MIMO MAC or BC
  - Outer Code:
    - ignores feedback
    - codes to achieve nofeedback capacity of super Gaussian MIMO MAC or BC

$\rightarrow$ Multi-letter expressions for $\mathcal{C}^{\mathrm{linfb}}_{\mathrm{MAC,CorrPower}}(h_1, h_2; \lambda, P)$ and $\mathcal{C}^{\mathrm{linfb}}_{\mathrm{BC}}(h_1, h_2, \lambda; P)$

Proof–Step 2: Dual Optimal Block-Feedback Schemes

- Identify pairs of MAC- and BC-block-feedback schemes that are dual

$\rightarrow$ For each MAC-params, $\exists$ BC-params achieving the same region, vice-versa.

If we choose

$$A_{i,\mathrm{BC}} = \bar{A}_{i,\mathrm{MAC}},$$

then super Gaussian MIMO MAC and BC have same nofeedback capacity.

($\bar{C}$: mirror image of C along counter-diagonal)

  - Proof: nofeedback MAC-BC duality & equivalence relations on capacity regions of super Gaussian MIMO channels

- Ozarow'84 $\longrightarrow$ Optimal BC parameters

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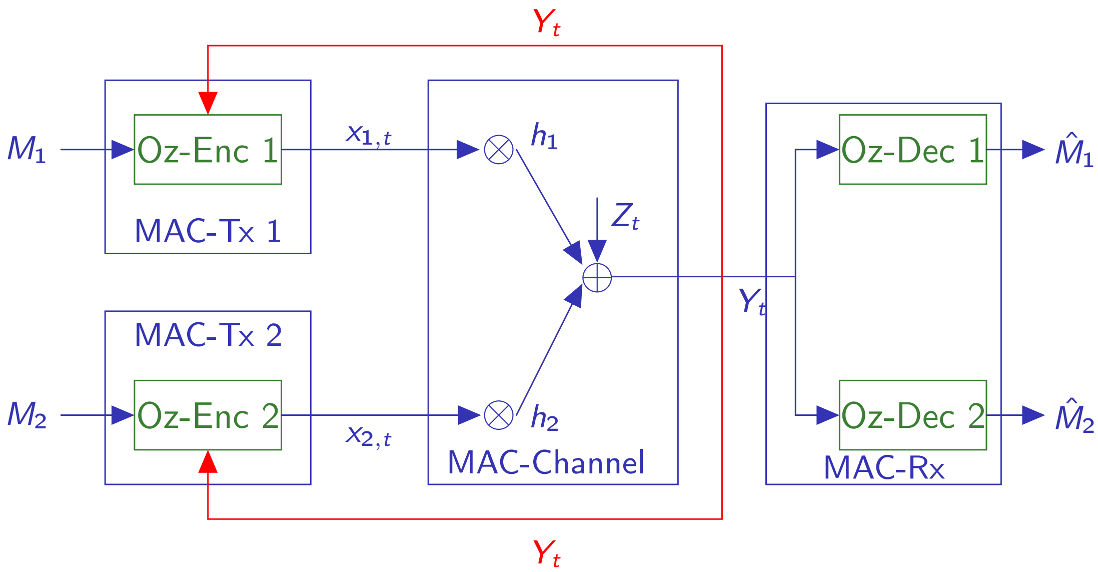
  - Proof: nofeedback MAC-BC duality & equivalence relations on capacity regions of super Gaussian MIMO channels

- Ozarow'84 $\longrightarrow$ Optimal BC parameters There is more than duality of achievable regions!

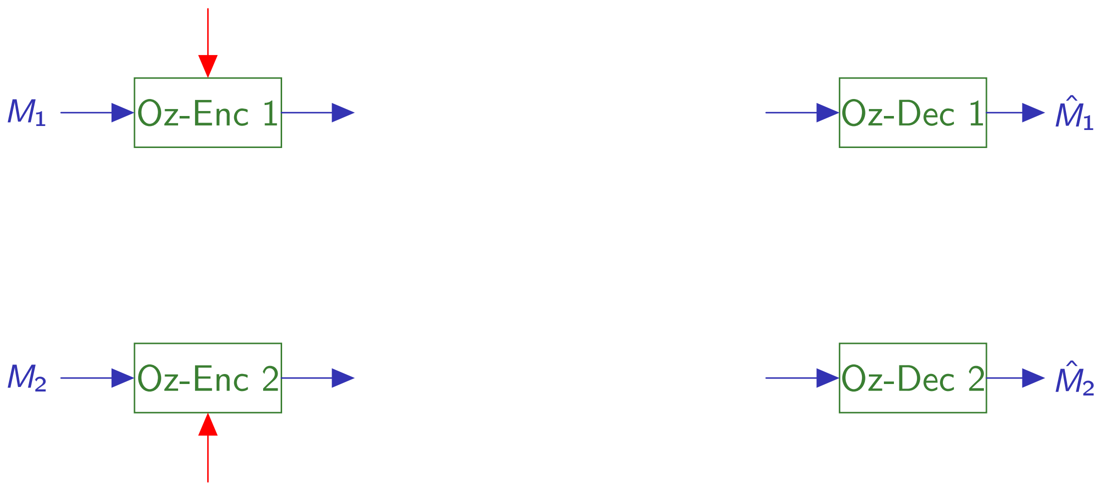
Main Results:
Constructive Sum-Rate Optimal BC-Scheme with Independent Noises

- Ozarow's MAC linear-feedback-sum-rate optimal scheme:



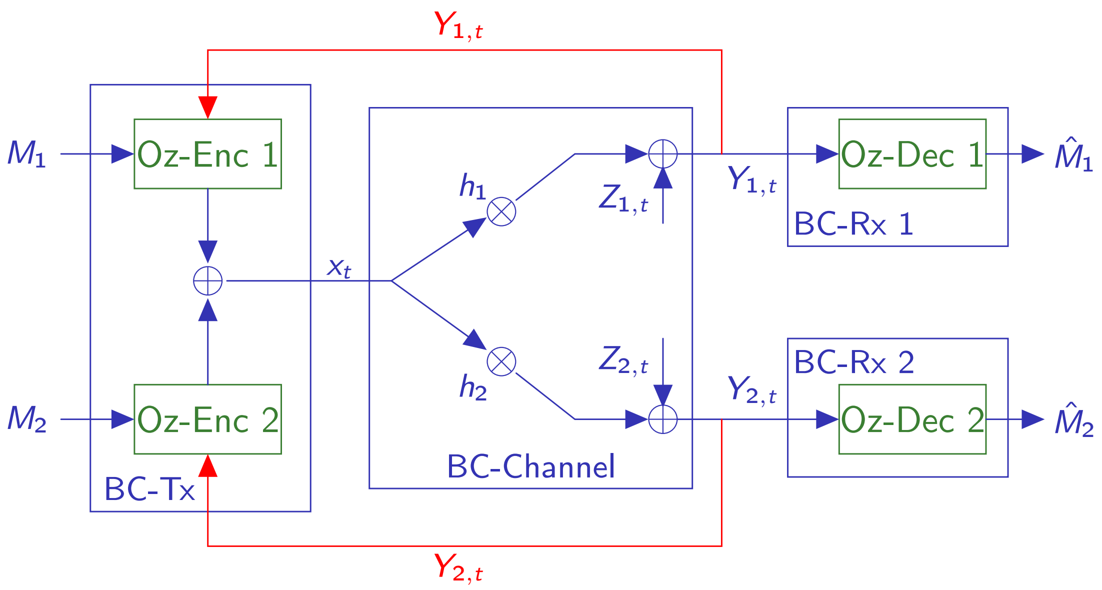
Main Results:
Constructive Sum-Rate Optimal BC-Scheme with Independent Noises

- We can use same encoding and decoding functions!



Main Results:
Constructive Sum-Rate Optimal BC-Scheme with Independent Noises

- Our new linear-feedback-sum-optimal BC-scheme:



Summary

- BC-noises correlation factor $\lambda \in [-1, 1]$
- MAC with "non-standard" sum-power constraint depending on λ

MAC-BC Duality with Correlated Noises at BC-Receivers

$$\mathcal{C}_{\text{BC}}^{\text{linfb}}(h_1, h_2, \lambda; P) = \mathcal{C}_{\text{BC}}^{\text{linfb}}(h_1, h_2; \lambda, P).$$

- Transfer MAC results to BC
  - (Ozarow'84) $\Longrightarrow$ Achievable region for BC with correlated noises

Constructive Sum-Rate Optimal BC-Scheme ($\lambda = 0$)

- Simple rearrangement Ozarow's MAC-encoders and MAC-decoders achieves $C_{\text{BC},\Sigma}^{\text{linfb}}$ for non-symmetric BC.