Lecture 4: Multiple Access Channels

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1 Discrete Multiple Access Channel

Definition 1 (Discrete Memoryless Multiple Access Channel (DM-MAC)). A DM-MAC is defined by a tuple $(X_1, X_2, Y, P_{Y|X_1X_2})$, where $X_1$ and $X_2$ are finite input alphabets, $Y$ is a finite output alphabet and $P_{Y|X_1X_2}$ is a conditional probability mass function such that for all $(x_1, x_2, y) \in X_1 \times X_2 \times Y$, $P_{Y|X_1X_2}(y|x_1, x_2) = P_{Y|X_1X_2}(y|x_1)P_{X_1}(x_1)P_{X_2}(x_2)$, where $P_{X_1}(x_1)$ and $P_{X_2}(x_2)$ are the corresponding probabilities of the input symbols $x_1$ and $x_2$.

Definition 2 (Codes for Multiple Access Channel (DM-MAC)). A code for a DM-MAC $(X_1, X_2, Y, P_{Y|X_1X_2})$ is determined by the tuple $(M_1, M_2, n, f_1^{(n)}, f_2^{(n)}, \phi^{(n)})$, where $f_1^{(n)}$ and $f_2^{(n)}$ are the encoding functions, such that for all $i \in \{1, 2\}$,

\[ f_i^{(n)} : \{1, 2, \ldots, M_i\} \to X_i^n \tag{1} \]

and $\phi^{(n)}$ is the decoding function, such that

\[ \phi^{(n)} : Y^n \to \{1, 2, \ldots, M_1\} \times \{1, 2, \ldots, M_2\}. \tag{2} \]

Definition 3 (Rate of a Code in the DM-MAC). The rate pair $(R_1, R_2)$ associated to a code $(M_1, M_2, n, f_1^{(n)}, f_2^{(n)}, \phi^{(n)})$ satisfies for all $i \in \{1, 2\}$,

\[ R_i = \frac{\log_2(M_i)}{n}. \tag{3} \]

Definition 4 (Probability of Error). The probability of error $P_e^{(n)}$ of a code $(M_1, M_2, n, f_1^{(n)}, f_2^{(n)}, \phi^{(n)})$ in the DM-MAC $(X_1, X_2, Y, P_{Y|X_1X_2})$ is

\[ P_e = \Pr [\phi^{(n)}(Y) \neq (W_1, W_2)]. \tag{4} \]
**Definition 5** (Achievable Rates). The pair \((R_1, R_2) \in \mathbb{R}^2_+\) is achievable in the DM-MAC \((\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1X_2})\) if there exists a sequence of tuples \(\left\{\left(2^nR_1, 2^nR_2, n, f_1^{(n)}, f_2^{(n)}, \phi^{(n)}\right)\right\}_{n=1}^\infty\) such that the error probability tends to zero as the blocklength \(n\) tends to infinity. That is,
\[
\limsup_{n \to \infty} P_e^{(n)} = 0.
\]

**Definition 6** (Capacity Region of the DM-MAC). The information capacity region \(\mathcal{C}_{\text{DM-MAC}} \subseteq \mathbb{R}^2\) of the DM-MAC \((\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1X_2})\) is the closure of all achievable information rate pairs \((R_1, R_2)\).

**Theorem 1** (Capacity Region of the DM-MAC). The capacity region \(\mathcal{C}_{\text{DM-MAC}}\) of the DM-MAC \((\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1X_2})\) is
\[
R_1 < I(X_1; Y|X_2),
R_2 < I(X_2; Y|X_1), \text{ and}
R_1 + R_2 < I(X_1, X_2; Y),
\]
for some \(P_{X_1} \in \Delta(\mathcal{X}_1)\) and \(P_{X_2} \in \Delta(\mathcal{X}_2)\).

**Theorem 2** (Capacity Region of the DM-MAC). The capacity region \(\mathcal{C}_{\text{DM-MAC}}\) of the DM-MAC \((\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1X_2})\) is a convex set of \(\mathbb{R}^2\).

**Theorem 3** (Achievable Region of the DM-MAC). The achievable region \(\mathcal{C}_{\text{DM-MAC}}\) of the DM-MAC \((\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1X_2})\) is
\[
R_1 < I(X_1; Y|X_2, Q),
R_2 < I(X_2; Y|X_1, Q), \text{ and}
R_1 + R_2 < I(X_1, X_2; Y|Q),
\]
for some joint distribution \(P_{QX_1X_2} = P_QP_{X_2|Q}P_{X_2|Q}\) and \(|Q| \leq 4\).

**Theorem 4** (Converse Region of the DM-MAC). The converse region \(\overline{\mathcal{C}}_{\text{DM-MAC}}\) of the DM-MAC \((\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1X_2})\) is
\[
R_1 < I(X_1; Y|X_2, Q),
R_2 < I(X_2; Y|X_1, Q), \text{ and}
R_1 + R_2 < I(X_1, X_2; Y|Q),
\]
for some joint distribution \(P_{QX_1X_2} = P_QP_{X_2|Q}P_{X_2|Q}\) and \(|Q| \leq 4\).

**Corollary 1** (Capacity Region of the DM-MAC). The achievable region \(\mathcal{C}_{\text{DM-MAC}}\), the converse region \(\overline{\mathcal{C}}_{\text{DM-MAC}}\), and the capacity region \(\mathcal{C}_{\text{DM-MAC}}\) of the DM-MAC satisfy
\[
\mathcal{C}_{\text{DM-MAC}} = \overline{\mathcal{C}}_{\text{DM-MAC}} = \mathcal{C}_{\text{DM-MAC}}.
\]
Consider the two-user memoryless Gaussian MAC (G-MAC) in Fig. 1. The goal of the communication is to convey the independent messages $M_1$ and $M_2$ from transmitters 1 and 2 to the common receiver using a blocklength of $n \in \mathbb{N}$ channel uses. The messages $W_1$ and $W_2$ are mutually independent and uniformly distributed over the sets $\mathcal{W}_1 = \{1, \ldots, 2^{nR_1}\}$ and $\mathcal{W}_2 = \{1, 2, \ldots, 2^{nR_2}\}$, where $R_1 = \frac{\log |\mathcal{W}_1|}{n}$ and $R_2 = \frac{\log |\mathcal{W}_2|}{n}$ denote the transmission rates. For all $i \in \{1, 2\}$, the channel inputs $X_i = (X_{i,1}, X_{i,2}, \ldots, X_{i,n}) \in \mathbb{R}^n$ are generated at the beginning of the transmission using the encoding functions

$$f_i^{(n)} : \mathcal{W}_i \to \mathbb{R}^n.$$  

That is $X_i = f_i^{(n)}(W_i)$ and moreover, the channel inputs $X_{i,1}, X_{i,2}, \ldots, X_{i,n}$ satisfy an expected average input power constraint:

$$\frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[X_{i,t}^2] \leq P_i, \quad (17)$$

where $P_i$ denotes the energy rate constraint of transmitter $i$. For all $t \in \{1, 2, \ldots, n\}$, the channel inputs $X_{1,t}$ and $X_{2,t}$ generate the channel output $Y_t$ according to a probability density function $f_{Y|X_1,X_2}$ for which the following holds

$$Y_t = h_1 X_{1,t} + h_2 X_{2,t} + Z_t, \quad (18)$$

where $h_1$ and $h_2$ are the corresponding constant non-negative channel coefficients from transmitter $i$ to the receiver. The noise terms $Z_1, Z_2, \ldots, Z_n$ are zero-mean unit-variance real Gaussian random variables.

The receiver produces an estimate $(\hat{W}_1^{(n)}, \hat{W}_2^{(n)}) = \Phi^{(n)}(Y)$ of the message-pair $(W_1, W_2)$ via a decoding function $\Phi^{(n)} : \mathbb{R}^n \to \mathcal{W}_1 \times \mathcal{W}_2$, and the average probability of error is

$$P_e^{(n)} = \Pr[(\hat{W}_1^{(n)}, \hat{W}_2^{(n)}) \neq (W_1, W_2)]. \quad (19)$$

The G-MAC is fully described by two parameters: the signal to noise ratios with respect to transmitter 1, denoted by SNR$_1$ and to transmitter 2, denoted by SNR$_2$. Note that given that the variance of the noise is 1, these parameters are defined as follows:

$$\text{SNR}_1 = P_1 \quad \text{and} \quad \text{SNR}_2 = P_2.$$  

$$\text{(20)} \quad \text{(21)}$$
**Theorem 5** (Capacity Region of the G-MAC). *The capacity region* $C_{G-MAC}$ *of the G-MAC with parameters* $\text{SNR}_1$ *and* $\text{SNR}_2$ *is*

\[
R_1 < \frac{1}{2} \log (1 + \text{SNR}_1), \quad (22)
\]

\[
R_2 < \frac{1}{2} \log (1 + \text{SNR}_2), \quad \text{and} \quad (23)
\]

\[
R_1 + R_2 < \frac{1}{2} \log (1 + \text{SNR}_1 + \text{SNR}_2). \quad (24)
\]