Lecture 4: Multiple Access Channels

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1 Discrete Multiple Access Channel

Definition 1 (Discrete Memoryless Multiple Access Channel (DM-MAC)). *A DM-MAC is defined by a tuple $\left(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1X_2}\right)$, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are finite input alphabets, $\mathcal{Y}$ is a finite output alphabet and $P_{Y|X_1X_2}$ is a conditional probability mass function such that for all $(x_1, x_2, y) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}$, $P_{X_1X_2Y}(x_1, x_2, y) = P_{Y|X_1X_2}(y|x_1, x_2)P_{X_1}(x_1)P_{X_2}(x_2)$, where $P_{X_1}(x_1)$ and $P_{X_2}(x_2)$ are the corresponding probabilities of the input symbols x_1 and x_2.*

Definition 2 (Codes for Multiple Access Channel (DM-MAC)). *A code for a DM-MAC $\left(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1X_2}\right)$ is determined by the tuple $\left(M_1, M_2, n, f_1^{(n)}, f_2^{(n)}, \phi^{(n)}\right)$, where $f_1^{(n)}$ and $f_2^{(n)}$ are the encoding functions, such that for all $i \in \{1, 2\}$,*

$$f_i^{(n)} \colon \{1, 2, \ldots, M_i\} \to \mathcal{X}_i^n \tag{1}$$

and $\phi^{(n)}$ is the decoding function, such that

$$\phi^{(n)} \colon \mathcal{Y}^n \to \{1, 2, \ldots, M_1\} \times \{1, 2, \ldots, M_2\}. \tag{2}$$

Definition 3 (Rate of a Code in the DM-MAC). *The rate pair (R_1, R_2) associated to a code $\left(M_1, M_2, n, f_1^{(n)}, f_2^{(n)}, \phi^{(n)}\right)$ satisfies for all $i \in \{1, 2\}$,*

$$R_i = \frac{\log_2(M_i)}{n}. \tag{3}$$

Definition 4 (Probability of Error). *The probability of error $P_{\mathrm{e}}^{(n)}$ of a code $\left(M_1, M_2, n, f_1^{(n)}, f_2^{(n)}, \phi^{(n)}\right)$ in the DM-MAC $\left(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1X_2}\right)$ is*

$$P_{\mathrm{e}} = \Pr\left[\phi^{(n)}\left(\boldsymbol{Y}\right) \neq (W_1, W_2)\right]. \tag{4}$$

Definition 5 (Achievable Rates). *The pair* $(R_1, R_2) \in \mathbb{R}_+^2$ *is achievable in the DM-MAC* $\left(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1X_2}\right)$ *if there exists a sequence of tuples* $\left\{\left(2^{nR_1}, 2^{nR_2}, n, f_1^{(n)}, f_2^{(n)}, \phi^{(n)}\right)\right\}_{n=1}^{\infty}$ *such that the error probability tends to zero as the blocklength* n *tends to infinity. That is,*

$$\limsup_{n\to\infty}\ P_{\mathrm{e}}^{(n)}{=}0. \tag{5}$$

Definition 6 (Capacity Region of the DM-MAC). *The information capacity region* $\mathcal{C}_{\mathrm{DM-MAC}} \subseteq \mathbb{R}_+^2$ *of the DM-MAC* $\left(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1X_2}\right)$ *is the closure of all achievable information rate pairs* (R_1, R_2).

Theorem 1 (Capacity Region of the DM-MAC). *The capacity region* $\mathcal{C}_{\mathrm{DM-MAC}}$ *of the DM-MAC* $\left(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1X_2}\right)$ *is*

$$\begin{aligned} R_1 \quad &< I(X_1;Y|X_2), && (6)\\ R_2 \quad &< I(X_2;Y|X_1), \text{ and} && (7)\\ R_1+R_2 &< I(X_1,X_2;Y), && (8)\end{aligned}$$

for some $P_{X_1} \in \triangle(\mathcal{X}_1)$ *and* $P_{X_2} \in \triangle(\mathcal{X}_2)$.

Theorem 2 (Capacity Region of the DM-MAC). *The capacity region* $\mathcal{C}_{\mathrm{DM-MAC}}$ *of the DM-MAC* $\left(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1X_2}\right)$ *is a convex set of* $\mathbb{R}^2$.

Theorem 3 (Achievable Region of the DM-MAC). *The achievable region* $\underline{\mathcal{C}}_{\mathrm{DM-MAC}}$ *of the DM-MAC* $\left(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1X_2}\right)$ *is*

$$\begin{aligned} R_1 \quad &< I(X_1;Y|X_2,Q), && (9)\\ R_2 \quad &< I(X_2;Y|X_1,Q), \text{ and} && (10)\\ R_1+R_2 &< I(X_1,X_2;Y|Q), && (11)\end{aligned}$$

for some joint distribution $P_{QX_1X_2} = P_Q P_{X_2|Q} P_{X_2|Q}$ *and* $|\mathcal{Q}| \leqslant 4$.

Theorem 4 (Converse Region of the DM-MAC). *The converse region* $\overline{\mathcal{C}}_{\mathrm{DM-MAC}}$ *of the DM-MAC* $\left(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1X_2}\right)$ *is*

$$\begin{aligned} R_1 \quad &< I(X_1;Y|X_2,Q), && (12)\\ R_2 \quad &< I(X_2;Y|X_1,Q), \text{ and} && (13)\\ R_1+R_2 &< I(X_1,X_2;Y|Q), && (14)\end{aligned}$$

for some joint distribution $P_{QX_1X_2} = P_Q P_{X_2|Q} P_{X_2|Q}$ *and* $|\mathcal{Q}| \leqslant 4$.

Corollary 1 (Capacity Region of the DM-MAC). *The achievable region* $\underline{\mathcal{C}}_{\mathrm{DM-MAC}}$, *the converse region* $\overline{\mathcal{C}}_{\mathrm{DM-MAC}}$ *and the capacity region* $\mathcal{C}_{\mathrm{DM-MAC}}$ *of the DM-MAC satisfy*

$$\underline{\mathcal{C}}_{\mathrm{DM-MAC}} = \overline{\mathcal{C}}_{\mathrm{DM-MAC}} = \mathcal{C}_{\mathrm{DM-MAC}}. \tag{15}$$

2 Gaussian Multiple Access Channel

Figure 1: Two-User Discret Memoryless Multiple Access Channel

Consider the two-user memoryless Gaussian MAC (G-MAC) in Fig. 1. The goal of the communication is to convey the independent messages M_1 and M_2 from transmitters 1 and 2 to the common receiver using a blocklength of $n \in \mathbb{N}$ channel uses. The messages W_1 and W_2 are mutually independent and uniformly distributed over the sets $\mathcal{W}_1 \triangleq \{1, \ldots, 2^{nR_1}\}$ and $\mathcal{W}_2 \triangleq \{1, 2, \ldots, 2^{nR_2}\}$, where $R_1 = \frac{\log_2 |\mathcal{W}_1|}{n}$ and $R_2 = \frac{\log_2 |\mathcal{W}_2|}{n}$ denote the transmission rates. For all $i \in \{1, 2\}$, the channel inputs $\boldsymbol{X}_i = (X_{i,1}, X_{i,2}, \ldots, X_{i,n}) \in \mathbb{R}^n$ are generated at the beginning of the transmission using the encoding functions

$$f_i^{(n)} \colon \mathcal{W}_i \to \mathbb{R}^n. \tag{16}$$

That is $\boldsymbol{X}_i = f_i^{(n)}(W_i)$ and moreover, the channel inputs $X_{i,1}, X_{i,2}, \ldots, X_{i,n}$ satisfy an expected average *input power constraint*:

$$\frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left[X_{i,t}^2\right] \leqslant P_i, \tag{17}$$

where P_i denotes the energy rate constraint of transmitter i. For all $t \in \{1, 2, \ldots, n\}$, the channel inputs $X_{1,t}$ and $X_{2,t}$ generate the channel output Y_t according to a probability density function $f_{Y|X_1X_2}$ for which the following holds

$$Y_t = h_1 X_{1,t} + h_2 X_{2,t} + Z_t, \tag{18}$$

where h_1 and h_2 are the corresponding constant non-negative channel coefficients from transmitter i to the receiver. The noise terms $Z_1, Z_2, \ldots, Z_n$ are zero-mean unit-variance real Gaussian random variables.
The receiver produces an estimate $(\hat{W}_1^{(n)}, \hat{W}_2^{(n)}) = \Phi^{(n)}(\boldsymbol{Y})$ of the message-pair (W_1, W_2) via a decoding function $\Phi^{(n)} \colon \mathbb{R}^n \to \mathcal{W}_1 \times \mathcal{W}_2$, and the average probability of error is

$$P_{\mathrm{e}}^{(n)} = \Pr\left[(\hat{W}_1, \hat{W}_2) \neq (W_1, W_2)\right]. \tag{19}$$

The G-MAC is fully described by two parameters: the signal to noise ratios with respect to transmitter 1, denoted by SNR_1 and to transmitter 2, denoted by SNR_2. Note that given that the variance of the noise is 1, these parameters are defined as follows:

$$\mathrm{SNR}_1 {=} P_1 \text{ and} \tag{20}$$

$$\mathrm{SNR}_2 {=} P_2. \tag{21}$$

Theorem 5 (Capacity Region of the G-MAC)**.** *The capacity region $\mathcal{C}_{\mathrm{G-MAC}}$ of the G-MAC with parameters* SNR_1 *and* SNR_2 *is*

$$R_1 < \frac{1}{2}\log\left(1+\mathrm{SNR}_1\right), \tag{22}$$

$$R_2 < \frac{1}{2}\log\left(1+\mathrm{SNR}_2\right), \text{ and} \tag{23}$$

$$R_1 + R_2 < \frac{1}{2}\log\left(1+\mathrm{SNR}_1+\mathrm{SNR}_2\right). \tag{24}$$