I Weak Typicality

• **Theorem 1** Let $X$ be a random variable $X$ defined on a countable set $X$ with pmf $P_X$. Let also $X = (X_1, X_2, \ldots, X_n)^T \in X^n$ be an $n$-dimensional random variable with pmf

$$P_X(x_1, x_2, \ldots, x_n) = \prod_{k=1}^{n} P_X(x_k), \quad (i)$$

for all $(x_1, x_2, \ldots, x_n) \in X^n$. Then, the random variable $X$ satisfies that for a large $n$, there always exists an $\epsilon > 0$ such that

$$\Pr \left[ \frac{1}{n} \log_2 \left( P_X(X) \right) - H(X) < \epsilon \right] = 1 - \epsilon. \quad (2)$$

• **Definition 1 (Weakly Typical Set)** The set of weakly typical outcomes of the $n$-dimensional random variable $X = (X_1, X_2, \ldots, X_n)^T$ whose pmf is given by (i), denoted by $T_X^{(n, \epsilon)}$, is

$$T_X^{(n, \epsilon)} = \{ x \in X^n : \left| \frac{1}{n} \log_2 (P_X(x)) - H(X) \right| < \epsilon \}. \quad (3)$$

• **Theorem 2** Let $T_X^{(n, \epsilon)}$ be the set of weakly typical outcomes of the $n$-dimensional random variable $X = (X_1, X_2, \ldots, X_n)^T$, with $\epsilon > 0$. Then, for $n$ sufficiently large, it holds that:

$$\forall x \in T_X^{(n, \epsilon)}, \quad 2^{-n(H(X)+\epsilon)} < P_X(x) < 2^{-n(H(X)-\epsilon)};$$

$$(1 - \epsilon)2^{n(H(X)-\epsilon)} < |T_X^{(n, \epsilon)}| < 2^{n(H(X)+\epsilon)} \quad \text{and} \quad (5)$$

$$1 - \epsilon < \sum_{x \in T_X^{(n, \epsilon)}} P_X(x) < 1. \quad (6)$$

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2 Weak Joint Typicality

- **Definition 2 (Weakly Joint Typical Set)** The set of weakly joint typical outcomes of the \((n \times 2)\)-dimensional random variable \((X, Y) = ((X_1, Y_1)^T, (X_2, Y_2)^T, \ldots, (X_n, Y_n)^T)^T\), whose pmf is given by

\[
P_{XY}(x, y) = \prod_{k=1}^{n} P_{XY}(x_k, y_k), \text{ for all } (x, y) \in (\mathcal{X} \times \mathcal{Y})^n, \tag{7}
\]

is

\[
\mathcal{T}_{X Y}^{(n, \epsilon)} = \left\{ (x, y) \in (\mathcal{X} \times \mathcal{Y})^n : \left| -\frac{1}{n} \log_2 (P_X (x)) - H(X) \right| < \epsilon, \quad -\frac{1}{n} \log_2 (P_Y (y)) - H(Y) \right| < \epsilon, \right. \quad \text{and} \quad \left. \left| -\frac{1}{n} \log_2 (P_{XY} (x, y)) - H(X, Y) \right| < \epsilon \right\}. \tag{8}
\]

- If \((x, y) \in \mathcal{T}_{X Y}^{(n, \epsilon)}\) then \(x \in \mathcal{T}_{X}^{(n, \epsilon)}\) and \(y \in \mathcal{T}_{Y}^{(n, \epsilon)}\).

- **Theorem 3** Let \(\epsilon > 0\) be arbitrarily small and \(n \in \mathbb{N}\) be sufficiently large. Let also \(\mathcal{T}_{X Y}^{(n, \epsilon)}\) be the set of weakly typical outcomes of the \((n \times 2)\)-dimensional random variable \((X, Y) = ((X_1, Y_1)^T, (X_2, Y_2)^T, \ldots, (X_n, Y_n)^T)^T\), whose pmf is given by (7). Then, it holds that:

\[
\forall (x, y) \in \mathcal{T}_{X Y}^{(n, \epsilon)}, \quad 2^{-n(H(X,Y)+\epsilon)} < P_{XY}(x,y) < 2^{-n(H(X,Y)-\epsilon)}, \tag{11}
\]

\[
2^{-n(H(X|Y)+2\epsilon)} < P_{X|Y}(x|y) < 2^{-n(H(X|Y)-2\epsilon)}, \tag{12}
\]

\[
(1 - \epsilon)2^{n(H(X,Y)-\epsilon)} < |\mathcal{T}_{X Y}^{(n,\epsilon)}| < 2^{n(H(X,Y)+\epsilon)} \quad \text{and} \quad (13)
\]

\[
1 - \epsilon < \sum_{(x,y) \in \mathcal{T}_{X Y}^{(n,\epsilon)}} P_{XY}(x,y) < 1. \tag{14}
\]

- **Theorem 4** Let \(\epsilon > 0\) be arbitrarily small and \(n \in \mathbb{N}\) be sufficiently large. Let also \(\mathcal{T}_{X Y}^{(n, \epsilon)}\) be the set of weakly typical outcomes of the \((n \times 2)\)-dimensional random variable \((X, Y)\), whose pmf is given by (7). Consider the \((n \times 2)\)-dimensional random variable \((V, W) \in (\mathcal{X} \times \mathcal{Y})^n\), such that \(\text{supp} (P_{XY}) = \text{supp} (P_{VW})\) with pmf given by

\[
P_{VW}(v, w) = \prod_{k=1}^{n} \left( \sum_{s \in \mathcal{Y}} P_{XY}(v_k, s) \right) \left( \sum_{t \in \mathcal{X}} P_{XY}(t, w_k) \right), \text{ for all } (v, w) \in (\mathcal{X} \times \mathcal{Y})^n.
\]
Then, it holds that:

\[ 2^{-n(I(X;Y)+3\epsilon)} < \sum_{(v,w) \in \mathcal{T}_{X|Y}^{(n,\epsilon)}} P_{VW}(v,w) < 2^{-n(I(X;Y)-3\epsilon)}. \] (15)

- **Theorem 5** Let \( X, Y \) and \( Z \) be three random variables such that \( X \) and \( Y \) are conditionally independent given \( Z \). Let \( X, Y \) and \( Z \) be three \( n \)-dimensional random variables with joint pmf

\[ P_{XYZ}(x, y, z) = \prod_{k=1}^{n} P_{X|Z}(x_k|z_k) P_{Y|Z}(y_k|z_k), \quad \forall (x, y, z) \in (\mathcal{X} \times \mathcal{Y} \times \mathcal{Z})^n \] (16)

Then, it holds that for all \( (x, y, z) \in (\mathcal{X} \times \mathcal{Y} \times \mathcal{Z})^n \),

\[ 2^{-n(I(X;Y|Z)+6\epsilon)} < P_{XYZ}(x, y, z) < 2^{-n(I(X;Y|Z)-6\epsilon)}. \] (17)

- All previous definitions and results can be extended to any number of variables.

## 3 Weak Conditional Typicality

- **Definition 3 (Weakly Typical Set Subject to Conditioning)** Let \( y = (y_1, \ldots, y_n) \) be an outcome such that \( y \in \mathcal{T}_{Y}^{(n,\epsilon)} \), with \( \epsilon > 0 \) and \( n \) sufficiently large. Then, the set of weakly typical outcomes of the \( n \)-dimensional random variable \( X = (X_1, X_2, \ldots, X_n) \) conditioning on the sequence \( y \), whose pmf is given by

\[ P_{X|Y}(x|y) = \prod_{k=1}^{n} P_{X|Y}(x_k|y_k), \quad \text{for all } x \in \mathcal{X}^n, \] (18)

is

\[ \mathcal{T}_{X|Y=y}^{(n,\epsilon)} = \left\{ x \in \mathcal{X}^n : \left| -\frac{1}{n} \log_2 \left( P_X(x) \right) - H(X) \right| < \epsilon, \quad \text{and} \right. \] (19)

\[ \left. \left| -\frac{1}{n} \log_2 \left( P_{XY}(x,y) \right) - H(X,Y) \right| < \epsilon \right\}. \] (20)

- **Theorem 6** Let \( \mathcal{T}_{X|Y=y}^{(n,\epsilon)} \) be the set of weakly typical outcomes of the \( n \)-dimensional random variable \( X \) conditioning on \( y \), with \( \epsilon > 0 \). Then, for \( n \) sufficiently large, it holds that:

\[ |\mathcal{T}_{X|Y=y}^{(n,\epsilon)}| < 2^n(H(X|Y)+2\epsilon) \] \hspace{1cm} (21)

\[ \mathbb{E}_Y \left[ |\mathcal{T}_{X|Y}^{(n,\epsilon)}| \right] > (1 - \epsilon)2^n(H(X|Y)-2\epsilon). \] (22)
4 Strong Typicality

• Consider a vector \( x = (x_1, \ldots, x_n)^T \in \mathcal{X}^n \), with \( n \in \mathbb{N} \) and \( \mathcal{X} \) a discrete and finite set. The type of the vector \( x \) is the function \( P_x : \mathcal{X} \to \mathbb{R}_+ \) defined as follows

\[
P_x(w) = \frac{1}{n} \sum_{i=1}^{n} 1_{\{x_i = w\}}.
\]

(23)

• The type \( P_x \) is the empirical pmf of the random variable \( X \) induced by the vector \( x \).

• Definition 4 (Strongly Typical Set) The strongly typical set with respect to a random variable \( X \) is

\[
T^{(n,\delta)}_X = \{ x \in \mathcal{X}^n : V(P_x, P_X) < \delta \}
\]

(24)

• Theorem 7 Let \( x \in T^{(n,\epsilon)}_X \). Then, \( x \in T^{(n,\delta)}_X \) for all \( \epsilon > 0 \) and \( \delta > 0 \).

• Theorem 8 Let \( T^{(n,\delta)}_X \) be the set of strongly typical \( n \)-tuples of outcomes of the random variable \( X \), with \( \delta > 0 \). Then, for \( n \) sufficiently large and \( \epsilon > 0 \), it holds that:

\[
\forall x \in T^{(n,\delta)}_X , \quad 2^{-n(H(X)+\epsilon)} < P_X(x) < 2^{-n(H(X)-\epsilon)},
\]

(25)

\[
(1-\delta)2^{n(H(X)-\epsilon)} < |T^{(n,\delta)}_X| < 2^{n(H(X)+\epsilon)} \quad \text{and}
\]

(26)

\[
1-\delta < \sum_{x \in T^{(n,\epsilon)}_X} P_X(x) < 1.
\]

(27)

5 Strong Joint Typicality

• Definition 5 (Strong Joint Typical Set) The set of strongly joint typical \( n \)-tuples of the random variables \( X \) and \( Y \) whose joint pmf is \( P_{XY} \) corresponds to the set:

\[
T^{(n,\delta)}_{XY} = \left\{ (x, y) \in (\mathcal{X} \times \mathcal{Y})^n : V(P_{xy}, P_{XY}) < \delta \right\}.
\]

(28)

• Theorem 9 Let \( (x, y) \in T^{(n,\delta)}_{XY} \). Then \( x \in T^{(n,\delta)}_X \) and \( y \in T^{(n,\delta)}_Y \).

• Theorem 10 Let \( X \) and \( Y \) be two random variables such that \( Y = g(X) \) with \( g \) any deterministic function. Hence, if \( x = (x_1, x_2, \ldots, x_n)^T \in T^{(n,\delta)}_X \) then,

\[
y = (g(x_1), g(x_2), \ldots, g(x_n))^T \in T^{(n,\delta)}_Y.
\]

(29)
**Theorem 11** Let \( \epsilon > 0 \) and \( \delta > 0 \) be arbitrarily small and \( n \in \mathbb{N} \) be sufficiently large. Let also \( T_{X,Y}^{(n,\delta)} \) be the set of strongly typical \( n \)-tuples of outcomes of the random variable pair \((X,Y)\), whose pmf is \( P_{XY} \). Then, it holds that:

\[
\forall (x,y) \in T_{X,Y}^{(n,\delta)}, \quad 2^{-n(H(X,Y)+\epsilon)} < P_{XY}(x,y) < 2^{-n(H(X,Y)-\epsilon)}, \quad (30)
\]

\[
2^{-n(H(X|Y))+2\epsilon} < P_{X|Y}(x|y) < 2^{-n(H(X|Y)-2\epsilon)}, \quad (31)
\]

\[
(1 - \delta)2^n(H(X,Y)-\epsilon) < |T_{X,Y}^{(n,\delta)}| < 2^n(H(X,Y)+\epsilon) \quad \text{and (32)}
\]

\[
1 - \delta < \sum_{(x,y) \in T_{X,Y}^{(n,\epsilon)}} P_{XY}(x,y) < 1. \quad (33)
\]

**Theorem 12** Let \( T_{X|Y=y}^{(n,\delta)} \) be the set of strongly typical \( n \)-tuples of outcomes of the random variable \( X \) conditioning on \( y \), with \( \delta > 0 \). Then, if \( |T_{X|Y=y}^{(n,\delta)}| \geq 1 \), it holds that:

\[
2^n(H(X|Y)+\epsilon) < |T_{X|Y=y}^{(n,\delta)}| < 2^n(H(X|Y)-\epsilon), \quad (35)
\]

with \( \epsilon > 0 \).

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**6 Strong Conditional Typicality**

**Definition 6 (Strong Typical Set Subject to Conditioning)** Let \( X \) and \( Y \) be two random variables with joint pmf \( P_{XY} \). Let \( y = (y_1, \ldots, y_n)^T \in T_{Y}^{(n,\delta)} \) be a strongly typical \( n \)-tuple of outcomes of the random variable \( Y \). The set of strongly typical \( n \)-tuples of outcomes of the random variable \( X \) conditioning on \( y \) is

\[
T_{X|Y=y}^{(n,\delta)} = \left\{ x \in X^n : V(P_x, P_{XY}) < \delta \right\}. \quad (34)
\]

**Theorem 12** Let \( T_{X|Y=y}^{(n,\delta)} \) be the set of strongly typical \( n \)-tuples of outcomes of the random variable \( X \) conditioning on \( y \), with \( \delta > 0 \). Then, if \( |T_{X|Y=y}^{(n,\delta)}| \geq 1 \), it holds that:

\[
2^n(H(X|Y)+\epsilon) < |T_{X|Y=y}^{(n,\delta)}| < 2^n(H(X|Y)-\epsilon), \quad (35)
\]

with \( \epsilon > 0 \).