Homework 4: Multiple Access Channels

Course: Network Information Theory (Fall 2016)
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1 Multiple Access Channels with per-Symbol Cost

Consider a DM-MAC $\left(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1X_2}\right)$ and let $b_i : \mathcal{X}_i \rightarrow \mathbb{R}_+$ be a cost function, for all $i \in \{1,2\}$. For all $x_i \in \mathcal{X}_i$, the cost payed by transmitter i for sending symbol x_i is $b_i(x_i)$.

For all $i \in \{1,2\}$, let $W_i \in \{1,2,\ldots,M_i\}$ be a uniformly distributed random variable representing the information source output associated with transmitter i. Given a code $\left(M_1, M_2, n, f_1^{(n)}, f_2^{(n)}, \phi^{(n)}\right)$, let $\boldsymbol{X}_i = f_i^{(n)}(W_i)$ be an n-dimensional random variable representing the channel inputs of transmitter i. Hence, the average cost payed by transmitter i when using this code is $\mathbb{E}_{\boldsymbol{X}_i}\left[\frac{1}{n}\sum_{t=1}^{n} b_i(X_{i,t})\right]$.

Definition 1 (Achievable Rates with Cost Constraints) *The pair $(R_1, R_2) \in \mathbb{R}_+^2$ is achievable in the DM-MAC $\left(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1X_2}\right)$ under average costs β_1 and β_2 if there exists a sequence of codes*

$$\left\{\left(2^{nR_1}, 2^{nR_2}, n, f_1^{(n)}, f_2^{(n)}, \phi^{(n)}\right)\right\}_{n=1}^{\infty} \tag{1}$$

such that the average cost for transmitter i, with $i \in \{1,2\}$, is upper bounded by β_i, i.e.,

$$\lim_{n\rightarrow\infty} \mathbb{E}_{\boldsymbol{X}_i}\left[\frac{1}{n}\sum_{t=1}^{n} b_i(X_{i,t})\right] \leqslant \beta_i, \tag{2}$$

and the error probability $P_{\mathrm{e}}^{(n)} = \Pr\left[\phi^{(n)}\left(\boldsymbol{Y}\right) \neq (W_1, W_2)\right]$ tends to zero as the blocklength n tends to infinity, i.e.,

$$\lim_{n\rightarrow\infty} P_{\mathrm{e}}^{(n)} = 0. \tag{3}$$

Following Def. 1, the capacity region of a DM-MAC $\left(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1X_2}\right)$ under average input costs β_1 and β_2 can be defined as follows:

Definition 2 (Capacity Region of the DM-MAC with input costs) *The information capacity region $\mathcal{C}_{\text{DM-MAC}}(\beta_1, \beta_2) \subseteq \mathbb{R}_+^2$ of the DM-MAC $\left(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1X_2}\right)$ with average input costs β_1 and β_2 is the closure of all achievable information rate pairs (R_1, R_2).*

2 Questions

1. Provide a full-characterization of $\mathcal{C}_{\text{DM-MAC}}(\beta_1, \beta_2) \subseteq \mathbb{R}_+^2$.
   Hint: Consider any two finite positive functions b_1 and b_2 and provide an achievable region and a converse region using the same steps presented in Lecture 4 and Lecture 5.

2. Show that for any two pairs (β_1, β_2) and (β_1', β_2'), with $\beta_1 \leqslant \beta_1'$ and $\beta_2 \leqslant \beta_2'$, it follows that $\mathcal{C}_{\text{DM-MAC}}(\beta_1, \beta_2) \subseteq \mathcal{C}_{\text{DM-MAC}}(\beta_1', \beta_2')$.

3. For all $i \in \{1, 2\}$, let $\beta_{i,\text{min}} = \min_{x \in \mathcal{X}_i} b_i(x)$ be the cost of the cheapest symbol. What can be said about $\mathcal{C}_{\text{DM-MAC}}(\beta_1, \beta_2)$ when $\beta_1 < \beta_{1,\text{min}}$ or $\beta_2 < \beta_{2,\text{min}}$?

4. Let $\beta_{1,\text{max}}$ be a positive real such that for all $\beta_1 > \beta_{1,\text{max}}$, it follows that $\mathcal{C}_{\text{DM-MAC}}(\beta_1, \beta_2) = \mathcal{C}_{\text{DM-MAC}}(\beta_{1,\text{max}}, \beta_2)$, with β_2 fixed and $\beta_2 > \beta_{2,\text{min}}$. What is the exact value of $\beta_{1,\text{max}}$?.

5. Consider a DM-MAC with binary inputs $\mathcal{X}_1 = \mathcal{X}_2 = \{0, 1\}$ and ternary outputs $\mathcal{Y} = \{0, 1, 2\}$ such that $Y = X_1 + X_2$. Let also $b_1(1) = b_2(1) = \gamma$ and $b_1(0) = b_2(0) = 0$. Denote by $\mathcal{C}(\beta_1, \beta_2) \subseteq \mathbb{R}_+^2$ the capacity region of this channel subject to the average input costs β_1 and β_2. Give a full characterization of $\mathcal{C}(\beta_1, \beta_2)$ for a fixed γ with $\beta_1 = \beta_2 = \beta$ and draw it.
   Hint: Use the general formulas found in Point 1 and calculate probability distributions P_{X_1} and P_{X_2} to maximize at least 3 corner points. The region $\mathcal{C}(\beta)$ is the convex hull of the regions induced by the three probability distribution mentioned above.