1 Multiple Access Channels with per-Symbol Cost

Consider a DM-MAC $(X_1, X_2, Y, P_{Y|X_1,X_2})$ and let $b_i : X_i \to \mathbb{R}_+$ be a cost function, for all $i \in \{1, 2\}$. For all $x_i \in X_i$, the cost payed by transmitter $i$ for sending symbol $x_i$ is $b_i(x_i)$.

For all $i \in \{1, 2\}$, let $W_i \in \{1, 2, \ldots, M_i\}$ be a uniformly distributed random variable representing the information source output associated with transmitter $i$. Given a code $(M_1, M_2, n, f_1^{(n)}, f_2^{(n)}, \phi^{(n)})$, let $X_i = f_i^{(n)}(W_i)$ be an $n$-dimensional random variable representing the channel inputs of transmitter $i$. Hence, the average cost payed by transmitter $i$ when using this code is $\mathbb{E}_{X_i} \left[ \frac{1}{n} \sum_{t=1}^{n} b_i(X_{i,t}) \right]$.

**Definition 1 (Achievable Rates with Cost Constraints)** The pair $(R_1, R_2) \in \mathbb{R}_+^2$ is achievable in the DM-MAC $(X_1, X_2, Y, P_{Y|X_1,X_2})$ under average costs $\beta_1$ and $\beta_2$ if there exists a sequence of codes

$$\left\{ \left( 2^{nR_1}, 2^{nR_2}, n, f_1^{(n)}, f_2^{(n)}, \phi^{(n)} \right) \right\}_{n=1}^{\infty}$$

such that the average cost for transmitter $i$, with $i \in \{1, 2\}$, is upper bounded by $\beta_i$, i.e.,

$$\lim_{n \to \infty} \mathbb{E}_{X_i} \left[ \frac{1}{n} \sum_{t=1}^{n} b_i(X_{i,t}) \right] \leq \beta_i,$$  \hspace{1cm} (2)

and the error probability $P_e^{(n)} = \Pr \left[ \phi^{(n)}(Y) \neq (W_1, W_2) \right]$ tends to zero as the blocklength $n$ tends to infinity, i.e.,

$$\lim_{n \to \infty} P_e^{(n)} = 0.$$  \hspace{1cm} (3)
Following Def. 1, the capacity region of a DM-MAC $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{\mathcal{Y}|\mathcal{X}_1\mathcal{X}_2})$ under average input costs $\beta_1$ and $\beta_2$ can be defined as follows:

**Definition 2 (Capacity Region of the DM-MAC with input costs)** The information capacity region $C_{DM-MAC}(\beta_1, \beta_2) \subseteq \mathbb{R}_+^2$ of the DM-MAC $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{\mathcal{Y}|\mathcal{X}_1\mathcal{X}_2})$ with average input costs $\beta_1$ and $\beta_2$ is the closure of all achievable information rate pairs $(R_1, R_2)$.

## 2 Questions

1. Provide a full-characterization of $C_{DM-MAC}(\beta_1, \beta_2) \subseteq \mathbb{R}_+^2$.
   **Hint:** Consider any two finite positive functions $b_1$ and $b_2$ and provide an achievable region and a converse region using the same steps presented in Lecture 4 and Lecture 5.

2. Show that for any two pairs $(\beta_1, \beta_2)$ and $(\beta'_1, \beta'_2)$, with $\beta_1 \leq \beta'_1$ and $\beta_2 \leq \beta'_2$, it follows that $C_{DM-MAC}(\beta_1, \beta_2) \subseteq C_{DM-MAC}(\beta'_1, \beta'_2)$.

3. For all $i \in \{1, 2\}$, let $\beta_{i,\min} = \min_{x \in \mathcal{X}_i} b_i(x)$ be the cost of the cheapest symbol. What can be said about $C_{DM-MAC}(\beta_1, \beta_2)$ when $\beta_1 < \beta_{1,\min}$ or $\beta_2 < \beta_{2,\min}$?

4. Let $\beta_{1,\max}$ be a positive real such that for all $\beta_1 > \beta_{1,\max}$, it follows that $C_{DM-MAC}(\beta_1, \beta_2) = C_{DM-MAC}(\beta_{1,\max}, \beta_2)$, with $\beta_2$ fixed and $\beta_2 > \beta_{2,\min}$. What is the exact value of $\beta_{1,\max}$?

5. Consider a DM-MAC with binary inputs $\mathcal{X}_1 = \mathcal{X}_2 = \{0, 1\}$ and ternary outputs $\mathcal{Y} = \{0, 1, 2\}$ such that $Y = X_1 + X_2$. Let also $b_1(1) = b_2(1) = \gamma$ and $b_1(0) = b_2(0) = 0$. Denote by $C(\beta_1, \beta_2) \subseteq \mathbb{R}_+^2$ the capacity region of this channel subject to the average input costs $\beta_1$ and $\beta_2$. Give a full characterization of $C(\beta_1, \beta_2)$ for a fixed $\gamma$ with $\beta_1 = \beta_2 = \beta$ and draw it.
   **Hint:** Use the general formulas found in Point 1 and calculate probability distributions $P_{X_1}$ and $P_{X_2}$ to maximize at least 3 corner points. The region $C(\beta)$ is the convex hull of the regions induced by the three probability distribution mentioned above.