Lecture 6: Relay channel

Course: Network Information Theory (Fall 2016)
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1 Discrete Relay Channel

Definition 1 (Discrete Memoryless Relay Channel (DM-RC)) *A DM-RC is defined by a tuple* $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_2, \mathcal{Y}_3, P_{Y_2Y_3|X_1X_2})$*, where* $\mathcal{X}_1$ *and* $\mathcal{X}_1$ *are finite input alphabets,* $\mathcal{Y}_2$ *and* $\mathcal{Y}_3$ *are finite output alphabets, and* $P_{Y_2Y_3|X_1X_2}$ *is a conditional probability mass function such that for all* $(x_1, x_2, y_2, y_3) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}_2 \times \mathcal{Y}_3$*,*
$P_{Y_2Y_3X_1X_2}(y_2, y_3, x_1, x_2) = P_{Y_2Y_3|X_1X_2}(y_2, y_3|x_1, x_2)P_{X_2X_1}(x_2, x_1)$*, where* $P_{X_1X_2}(x_1, x_2)$ *is the joint probability of the input symbols* (x_1, x_2)*.*

- Note : As a first sight, it may seem that x_1 and x_2 should be chosen independently, $P_{X_2X_1}(x_2, x_1) = P_{X_2}(x_2) \times P_{X_1}(x_1)$ to avoid external coordiniation. However, some dependence may be introduced thanks to the past symbols transmitted on the channel $X_1 \to Y_2$.

Definition 2 (Code for Relay Channel (DM-RC)) *:*
A code for a DM-RC $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_2, \mathcal{Y}_3, P_{Y_2Y_3|X_1X_2})$ *is defined by the tuple* $\left(M, n, f^{(n)}, \left\{g_k^{(n)}\right\}_{k=1}^n, \phi^{(n)}\right)$*, where* $f^{(n)}$ *is the encoding function such that*

$$f^{(n)} : \{1, 2, \ldots, M\} \to \mathcal{X}_1^{(n)}, \tag{1}$$

$\left\{g_k^{(n)}\right\}_{k=1}^n$ *is a set of relay functions such that*

$$g_k^{(n)} : \mathcal{Y}_2^{k-1} \to \mathcal{X}_2, \tag{2}$$

and are used to encode the kth *input symbol* $X_2(k) = g_k^{(n)}\left(Y_{2,1}, Y_{2,2}, \ldots, Y_{2,k-1}\right)$*,*
and $\phi^{(n)}$ *is the decoding function such that*

$$\phi^{(n)} : \mathcal{Y}_3^{(n)} \to \{1, 2, \ldots, M\}. \tag{3}$$

- Note that the relay encoding functions actually form part of the definition of the relay channel and impose causality condition at the relay.

Definition 3 (Rate of a Code in the DM-RC) *The rate R associated to a code $\left(M, n, f^{(n)}, \left\{g_k^{(n)}\right\}_{k=1}^{n}, \phi^{(n)}\right)$ satisfies*

$$R = \frac{\log_2(M)}{n}. \tag{4}$$

Definition 4 (Probability of Error) *The individual probabilities of error $\mathrm{P_e}^{(n)}$ of a code $\left(M, n, f^{(n)}, \left\{g_k^{(n)}\right\}_{k=1}^{n}, \phi^{(n)}\right)$ is*

$$\mathrm{P_e}^{(n)} = \mathrm{Pr}\left[\phi^{(n)}(\boldsymbol{Y}) \neq (W)\right], \tag{5}$$

where $W \in \{1, 2, \ldots, M\}$ is the index of the transmitted message.

Definition 5 (Achievable Rates) *:*
The rate $R \in \mathbb{R}_+$ is achievable in the DM-RC $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_2, \mathcal{Y}_3, P_{Y_2Y_3|X_1X_2})$ if there exists a sequence of tuples $\left\{\left(2^{nR}, n, f^{(n)}, \left\{g_k^{(n)}\right\}_{k=1}^{n}, \phi^{(n)}\right)\right\}_{n=1}^{\infty}$ such that the error probability $\mathrm{P_e}^{(n)}$ tends to zero as the blocklength n tends to infinity. That is,

$$\lim_{n\to\infty} \mathrm{sup}\mathrm{P_e}^{(n)} = 0. \tag{6}$$

Definition 6 (Capacity of the DM-RC) *The information capacity $\mathcal{C}_{DM-RC}$ is the highest achievable rate R.*

- Note: the capacity $\mathcal{C}_{DM-RC}$ of the DM-RC is not known in the general case.

Theorem 1 (CutSet Outer Bound) *The capacity $\mathcal{C}_{DM-RC}$ of the DM-RC $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_2, \mathcal{Y}_3, P_{Y_2Y_3|X_1X_2})$ is upper bounded by*

$$R < \max_{P_{X_1X_2}} \min\left(I(X_1, X_2; Y_3), I(X_1; Y_2, Y_3|X_2)\right). \tag{7}$$

Theorem 2 (Direct Transmission Lower Bound) *The capacity of the DM-RC $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_2, \mathcal{Y}_3, P_{Y_2Y_3|X_1X_2})$ is lower bounded by the capacity of the point-to-point DMC $(\mathcal{X}_1, \mathcal{Y}_3, P_{Y_3|X_1X_2=x_2})$ from the sender to the the receiver conditioned on some relay state x_2 defined as*

$$R \leq \max_{p_{X_1}, x_2} I(X_1; Y_3|X_2 = x_2). \tag{8}$$

- In this direct transmission coding scheme, the relay transmits the symbol that is the most favorable for a direct transmission from the transmitter to the receiver.
- This bound is tight for the *reversly degraded DM-RC*, in which $P_{Y_2Y_3|X_2X_1} = P_{Y_3|X_2X_1}P_{Y_2|Y_3X_2}$.
- But this bound may be not tight and is only an inner bound if the DM-RC is only *stochastically* reversely degraded.

Theorem 3 (Multihop Lower Bound) *The capacity of the DM-RC* $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_2, \mathcal{Y}_3, P_{Y_2Y_3|X_1X_2})$ *is lower bounded by*

$$C \geq \max_{P_{X_1}, P_{X_2}} \min\left(I(X_2;Y_3); I(X_1;Y_2|X_2)\right). \tag{9}$$

- This bound is achievable with a multihop transmission.
- In a multihop relaying scheme, a block coding transmission is used. The relay recovers the message received from the sender in each block and retransmits it in the following block.
- This bound is tight for a *cascade of two DM-RC*, defined such that $P_{Y_2Y_3|X_1X_2} = P_{Y_2|X_1}P_{Y_3|X_2}$.

Theorem 4 (Coherent Multihop Lower Bound) *The capacity of the DM-RC* $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_2, \mathcal{Y}_3, P_{Y_2Y_3|X_1X_2})$ *is lower bounded by*

$$C \geq \max_{P_{X_1,X_2}} \min\left(I(X_2;Y_3); I(X_1;Y_2|X_2)\right). \tag{10}$$

- this bound is achievable with a coherent multihop transmission scheme.
- In a coherent multihop relaying scheme, the relay recovers the message received from the sender in each block and retransmits it in the following block.

Theorem 5 (Decode and Forward Lower Bound) *The capacity of the DM-RC* $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_2, \mathcal{Y}_3, P_{Y_2Y_3|X_1X_2})$ *is lower bounded by*

$$C \leq \max_{p_{X_1,X_2}} \min\left(I(X_1,X_2;Y_3); I(X_1;Y_2|X_2)\right). \tag{11}$$

- This scheme improves the coherent multihop lower bound by letting the receiver decode simultaneously the messages sent by the sender and the relay.

- This bound is tight for the *degraded DM-RC*, in which $P_{Y_2Y_3|X_2X_1} = P_{Y_2|X_2X_1}P_{Y_3|Y_2X_2}$.
- This bound may not be tight and is only an inner bound if the DM-RC is stochastically degraded.

Theorem 6 (Partial Decode and Forward Lower Bound) *The capacity of the DM-RC* $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_2, \mathcal{Y}_3, P_{Y_2Y_3|X_1X_2})$ *is lower bounded by*

$$R \leq \max_{p_{UX_1X_2}} \min\left(I(X_1, X_2; Y_3), I(U; Y_2|X_2) + I(X_1; Y_3|X_2, U)\right), \tag{12}$$

where $|\mathcal{U}| \leq |\mathcal{X}_1| \cdot |\mathcal{X}_2|$.

- This scheme improves the decode and forward (DF) lower bound in some cases by letting the relay to decode a part of the message only.
- This scheme improves the DF bound when the capacity of the source-relay link is lower than the capacity of the source-destination link.

Theorem 7 (Compress and Forward Lower Bound) *The capacity of the relay channel is lower bounded as*

$$R \leq \max\min\left(I(X_1, X_2; Y_3) - I(Y_2; \hat{Y}_2|X_1, X_2, Y_3), I(X_1; \hat{Y}_2, Y_3|X_2)\right). \tag{13}$$

where the maximum is over all conditional pmfs $P_{X_1}P_{X_2}P_{\hat{Y}_2|X_2Y_2}$ *with* $|\hat{\mathcal{Y}}| \leq |\mathcal{X}_2| \cdot |\mathcal{Y}_2| + 1$.

- This scheme aims at improving the use of the relay-destination channel when it is a bottleneck.
- It improves the DF bound when the capacity of the relay-destination link is low.
- Wyner-Ziv coding is used.

2 Gaussian Relay Channel

The single relay with additive white Gaussian noise RC model is given by the following outputs:

$$Y_2 = g_{21}X_1 + Z_2, \tag{14}$$
$$Y_3 = g_{31}X_1 + g_{32}X_2 + Z_3, \tag{15}$$

where g_{21}, g_{31} and g_{32} are positive real numbers and $Z_i \sim \mathcal{N}(0,1)$ with $i \in \{1,2\}$, are mutually independent.

Thus, at time t, the channel outputs are:

$$Y_{2t} = g_{21}X_{1t} + Z_{2t}, \tag{16}$$
$$Y_{3t} = g_{31}X_{1t} + g_{32}X_{2t} + Z_{3t}. \tag{17}$$

The goal of the communication is to convey a messages M from the sender to the receiver using a blocklength of n channel uses with the help of the relay. The message index W is uniformly distributed over the sets $\mathcal{W} \triangleq \left\{1, \ldots, 2^{nR}\right\}$. The channel input $\boldsymbol{X_1} = (X_{11}, X_{12}, \ldots, X_{1n}) \in \mathbb{R}^n$ is generated using the encoding function:

$$f^{(n)} : \mathcal{W} \to \mathbb{R}^n. \tag{18}$$

The transmission is constrained by an average input power constraint $P_1 \in \mathbb{R}^+$:

$$\frac{1}{n}\sum_{t=1}^{n} \mathbb{E}_W\left[X_{1t}^2\right] \leq P_1 \quad . \tag{19}$$

The channel input at the relay $\boldsymbol{X_2} = (X_{21}, 22, \ldots, X_{2n}) \in \mathbb{R}^n$ is generated iteratively using the set of encoding functions:

$$g_k^{(n)} : \mathcal{X}_2^{k-1} \to \mathbb{R}, \quad \forall k \in [1, 2, \ldots, N], \tag{20}$$

and the transmission is constrained by an average input power constraint $P_2 \in \mathbb{R}^+$:

$$\frac{1}{n}\sum_{t=1}^{n} \mathbb{E}_W\left[X_{2t}^2\right] \leq P_2 \quad . \tag{21}$$

The Gaussian-RC is fully described by three SNR parameters

$$S_2 = \mathrm{SNR}_2 = g_{21}^2 P_1, \tag{22}$$
$$S_{31} = \mathrm{SNR}_{31} = g_{31}^2 P_1, \tag{23}$$
$$S_{32} = \mathrm{SNR}_{32} = g_{32}^2 P_2. \tag{24}$$

Theorem 8 (Upper Bound of the Gaussian-RC) *An upper bound of the Gaussian-RC is obtained when X_1 and X_2 are jointly Gaussian and provides*

$$R \leq \max_{0 \leq \rho \leq 1} \min\left\{C_g\left(S_{eq,1}\right), C_g\left(S_{eq,2}\right)\right\}, \tag{25}$$

with $C_g(SNR)$ the capacity of a point to point (P2P) Gaussian channel (see course 3) and

$$S_{eq,1} = S_{31} + S_{32} + 2\rho\sqrt{S_{31}S_{32}}, \tag{26}$$
$$S_{eq,2} = (1-\rho^2)(S_{31} + S_2), \tag{27}$$

where $\rho = \frac{\mathbb{E}[X_1X_2]}{\sqrt{\mathbb{E}[X_1^2]\mathbb{E}[X_2^2]}}$ is the correlation coefficient.

Theorem 9 (Lower Bounds of the Gaussian-RC) *The different lower bounds of the DM-RC can be computed with Gaussian input X_1, X_2 under the power constraints* (19) *and* (21). *The corresponding achievable rates follow*

- *Direct Transmission Lower Bound*

$$R \leq C_g(S_{31}). \tag{28}$$

- *Multihop Lower Bound*

$$R \leq \min\left\{C_g(S_{21}), C_g\left(\frac{S_{32}}{S_{31}+1}\right)\right\}. \tag{29}$$

- *Decode and Forward Lower Bound*

$$R \leq \max_{0\leq\rho\leq 1} \min\left\{C_g(S_{eq,1}), C_g\left(S_{eq,2}\right)\right\}, \tag{30}$$

  with

$$S_{eq,1} = S_{31} + S_{32} + 2\rho\sqrt{S_{31}S_{32}}, \tag{31}$$
$$S_{eq,2} = (1-\rho^2)S_2. \tag{32}$$

- *Compress and Forward Lower Bound*

$$R \leq C_g(S_{eq}) \tag{33}$$

  with

$$S_{eq} = S_{31} + \frac{S_2S_{32}}{S_{31}+S_2+S_{32}+1} \tag{34}$$

The former Gaussian-RC is called full duplex because the relay is able to receive and transmit simultaneously. When this is not the case, the relay alternates reception and transmission phases and the corresponding channel is called the half-duplex Gaussian-RC.