Lecture 5: Broadcast channel

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1 Discrete Broadcast Channel

Definition 1 (Discrete Memoryless Broadcast Channel (DM-BC)) *A DM-BC is defined by a tuple $(\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2, P_{Y_1Y_2|X})$, where $\mathcal{X}$ is a finite input alphabet, $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are finite output alphabets, and $P_{Y_1Y_2|X}$ is a conditional probability mass function such that for all $(x, y_1, y_2) \in \mathcal{X} \times \mathcal{Y}_1 \times \mathcal{Y}_2$, $P_{Y_1Y_2X}(y_1, y_2, x) = P_{Y_1Y_2|X}(y_1, y_2|x)P_X(x)$, where $P_X(x)$ is the probability of the input symbol x.*

Definition 2 (Code for Broadcast Channel (DM-BC)) *A code for a DM-BC $(\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2, P_{Y_1Y_2|X})$ is defined by the tuple $\left(M_0, M_1, M_2, n, f^{(n)}, \phi_1^{(n)}, \phi_2^{(n)}\right)$, where $f^{(n)}$ is the encoding function such that*

$$f^{(n)} : \{1, 2, \ldots, M_0\} \times \{1, 2, \ldots, M_1\} \times \{1, 2, \ldots, M_2\} \to \mathcal{X}^{(n)}, \tag{1}$$

and $\phi_i^{(n)}$ are the decoding functions such that for all $i \in \{1, 2\}$

$$\phi_i^{(n)} : \mathcal{Y}_i^{(n)} \to \{1, 2, \ldots, M_0\} \times \{1, 2, \ldots, M_i\}. \tag{2}$$

When $M_0 = 1$, the channel is called a DM-BC with private messages only.

Definition 3 (Rates of a Code in the DM-BC) *The rate tuple (R_0, R_1, R_2) associated to a code $\left(M_0, M_1, M_2, n, f^{(n)}, \phi_1^{(n)}, \phi_2^{(n)}\right)$ satisfies for all $i \in \{0, 1, 2\}$,*

$$R_i = \frac{\log_2(M_i)}{n}. \tag{3}$$

Definition 4 (Probability of Error) *The individual probabilities of error* ${\rm P_{e,i}}^{(n)}$ *of a code* $\left(M_0, M_1, M_2, n, f^{(n)}, \phi_1^{(n)}, \phi_2^{(n)}\right)$ *are*

$$
{\rm P_{ei}}^{(n)} = \Pr\left[\phi_i^{(n)}(\boldsymbol{Y}_i) \neq (W_0, W_i)\right], \tag{4}
$$

where W_1, W_2 *are the indices of the transmitted messages.*

Definition 5 (Achievable Rates) *The tuple* $(R_0, R_1, R_2) \in \mathbb{R}_+^3$ *is achievable in the DM-BC* $(\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2, P_{Y_1Y_2|X})$ *if there exists a sequence of tuples* $\left\{\left(2^{nR_0}, 2^{nR_1}, 2^{nR_2}, n, f^{(n)}, \phi_1^{(n)}, \phi_2^{(n)}\right)\right\}_{n=1}^{\infty}$ *such that the union of the individual error probabilities* ${\rm P_e}^{(n)} \leq {\rm P_{e,1}}^{(n)} + {\rm P_{e,2}}^{(n)}$ *tends to zero as the blocklength* n *tends to infinity. That is,*

$$
\lim_{n\to\infty} \sup {\rm P_e}^{(n)} = 0. \tag{5}
$$

Definition 6 (Capacity Region of the DM-BC) *The information capacity region* $\mathcal{C}_{DM-BC} \subseteq \mathbb{R}_+^3$ *of the DM-BC* $(\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2, P_{Y_1Y_2|X})$ *is the closure of all achievable information tuple* (R_0, R_1, R_2).

- Note: the capacity region $\mathcal{C}_{DM-BC}$ of the DM-BC is not known in the general case.

In the following the results are restricted to the DM-BC with private messages only, for simplicity purpose.

Theorem 1 (CutSet Outer Bound) *The capacity region* $\mathcal{C}_{DM-BC}$ *of the DM-BC* $(\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2, P_{Y_1Y_2|X})$ *is upper bounded by*

$$
\begin{aligned}
R_1 &< I(X;Y_1), && (6)\\
R_2 &< I(X;Y_2), && (7)\\
R_1 + R_2 &< I(X;Y_1,Y_2), && (8)
\end{aligned}
$$

Theorem 2 (Superposition Coding Inner Bound (with private messages only)) *A rate pair* (R_1, R_2) *is achievable for the DM-BC* $(\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2, P_{Y_1Y_2|X})$ *if*

$$
\begin{aligned}
R_1 &< I(X;Y_1|U), && (9)\\
R_2 &< I(U;Y_2), && (10)\\
R_1 + R_2 &< I(X;Y_1), && (11)
\end{aligned}
$$

for some joint distribution $P_{UX} = P_U P_{X|U}$.

Theorem 3 (Marton's Inner Bound (with private messages only)) *A rate pair* (R_1, R_2) *is achievable for the DM-BC* $(\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2, P_{Y_1Y_2|X})$ *if*

$$R_1 < I(U_1; Y_1), \tag{12}$$
$$R_2 < I(U_2; Y_2), \tag{13}$$
$$R_1 + R_2 < I(U_1; Y_1) + I(U_2; Y_2) - I(U_1; U_2), \tag{14}$$

for some joint distribution $P_{U_1U_2}$ *and a deterministic function* $x(u_1, u_2)$.

2 Degraded BC

Definition 7 (Physically Degraded BC) *A DM-BC* $(\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2, P_{Y_1Y_2|X})$ *is said to be* physically degraded *if*

$$P_{Y_1Y_2|X} = P_{Y_1|X} P_{Y_2|Y_1}, \tag{15}$$

- i.e. $X \to Y_1 \to Y_2$ forms a Markov chain.

Definition 8 (Stochastically Degraded BC) *A DM-BC* $(\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2, P_{Y_1Y_2|X})$ *is said to be* stochastically degraded *or simply degraded, if there exists a random variable* $\tilde{Y}_1$ *such that* $\tilde{Y}_1|X \sim P_{Y_1|X}$ *and* $X \to \tilde{Y}_1 \to Y_2$ *form a Markov chain.*

$$P_{\tilde{Y}_1Y_2|X} = P_{\tilde{Y}_1|X} P_{Y_2|\tilde{Y}_1}, \tag{16}$$

Theorem 4 (Achievable Region of the Stochastically Degraded DM-BC) *The achievable region* $\underline{\mathcal{C}}_{DM-BC}$ *of the degraded DM-BC* $(\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2, P_{Y_1Y_2|X})$ *is*

$$R_1 < I(X; Y_1|U), \tag{17}$$
$$R_2 < I(U; Y_2), \tag{18}$$

for some joint distribution P_{UX}.

Theorem 5 (Converse Region of the Stochastically Degraded DM-BC) *The converse region* $\overline{\mathcal{C}}_{DM-BC}$ *of the degraded DM-BC* $(\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2, P_{Y_1Y_2|X})$ *is*

$$R_1 \le I(X; Y_1|U), \tag{19}$$
$$R_2 \le I(U; Y_2), \tag{20}$$

for some joint distribution P_{UX} *and where* $|\mathcal{U}| \le \min\{|\mathcal{X}|, |\mathcal{Y}_1|, |\mathcal{Y}_2|\} + 1$.

Corollary 1 (Capacity Region of the Stochastically Degraded DM-BC) *The capacity region* $\mathcal{C}_{DM-BC}$ *of the degraded DM-BC* $(\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2, P_{Y_1Y_2|X})$ *satisfies:*

$$\mathcal{C}_{DM-BC} = \underline{\mathcal{C}}_{DM-BC} = \overline{\mathcal{C}}_{DM-BC}. \tag{21}$$

3 Gaussian Broadcast Channel

The two receivers discrete-time additive white Gaussian noise BC model is given by the following outputs:

$$Y_1 = g_1 \times X + Z_1, \tag{22}$$
$$Y_2 = g_2 \times X + Z_2, \tag{23}$$

where g_1 and g_2 are positive real numbers and $Z_i \sim \mathcal{N}(0,1)$, with $i \in \{1,2\}$ are mutually independent. Assume without loss of generality that $|g_1| > |g_2|$.

Thus, at time t, the channel outputs are:

$$Y_{1t} = g_1 \times X_t + Z_{1t}, \tag{24}$$
$$Y_{2t} = g_2 \times X_t + Z_{2t}. \tag{25}$$

For notational convenience, we consider the equivalent Gaussian BC

$$Y_1 = X + Z_1', \tag{26}$$
$$Y_2 = X + Z_2', \tag{27}$$

where $Z_i' \sim \mathcal{N}(0, N_i = 1/g_i^2)$, for $i \in \{1,2\}$.

The goal of the communication is to convey two independent messages M_1 and M_2 from the transmitter to two receivers using a blocklength of n channel uses. The message indexes W_1 and W_2 are mutually independent and uniformly distributed over the sets $\mathcal{W}_1 \triangleq \left\{1, \ldots, 2^{nR_1}\right\}$ and $\mathcal{W}_2 \triangleq \left\{1, \ldots, 2^{nR_2}\right\}$. The channel input $\boldsymbol{X} = (X_1, X_2, \ldots, X_n) \in \mathbb{R}^n$ is generated using the encoding function:

$$f^{(n)} : \mathcal{W}_1 \times \mathcal{W}_2 \to \mathbb{R}^n. \tag{28}$$

The transmission is constrained by an average input power constraint:

$$\frac{1}{n} \sum_{t=1}^{n} E_{W_1 W_2} \left[X_t^2\right] \leq P. \tag{29}$$

The Gaussian-BC is fully described by two SNR parameters

$$\mathrm{SNR}_1 = \frac{P}{N_1}, \tag{30}$$
$$\mathrm{SNR}_2 = \frac{P}{N_2}. \tag{31}$$

The transmitter uses two independent variables U, V with pdf $f_U \sim \mathcal{N}(0, \alpha P)$ and $f_V \sim \mathcal{N}(0, \bar{\alpha} P)$ where $\alpha \in [0,1]$ and $\bar{\alpha} = 1 - \alpha$.

The transmitter generates two codebooks :

- Randomly and independently generate 2^{nR_2} sequences $\boldsymbol{U}(W_2)$ according to f_U.
- Randomly and independently generate 2^{nR_1} sequences $\boldsymbol{V}(W_1)$ according to f_V.

Then $\boldsymbol{X}$ is generated according to $\boldsymbol{X}(W_1, W_2) = \boldsymbol{U}(W_2) + \boldsymbol{V}(W_1)$.

Receiver 1 uses successive decoding: it firstly decodes $\hat{W}_2 = \phi_{12}^n(\boldsymbol{Y}_1)$. Then it generates $\boldsymbol{U}$ and decodes $\hat{W}_1 = \phi_{11}^n(\boldsymbol{Y}_1 - \boldsymbol{U})$. Receiver 2 decodes directly $\hat{W}_2 = \phi_2^n(\boldsymbol{Y}_2)$.

Theorem 6 (Capacity Region of the Gaussian-BC) *The capacity region $\mathcal{C}_{G-BC}$ of the Gaussian-BC is*

$$R_1 < \frac{1}{2} \log \left(1 + \alpha \mathrm{SNR}_1\right), \tag{32}$$

$$R_2 < \frac{1}{2} \log \left(1 + \frac{\bar{\alpha}\mathrm{SNR}_2}{1 + \alpha \mathrm{SNR}_2}\right), \tag{33}$$

for some $\alpha \in [0,1]$.